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# Computational Methodology for Aeroacoustic Simulation Using a Volume Penalization Method and CAD Generated Geometry

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**Computational Methodology for Aeroacoustic Simulation  
Using a Volume Penalization Method and CAD Generated  
Geometry**

by

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B.S., Mechanical Engineering, Washington State University, 2008

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Computational Methodology for Aeroacoustic Simulation Using a Volume Penalization Method  
and CAD Generated Geometry  
written by Eric Brown-Dymkoski  
has been approved for the Department of Mechanical Engineering

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Oleg V. Vasilyev

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Prof. Daven Henze

Date \_\_\_\_\_

The final copy of this thesis has been examined by the signatories, and we find that both the content and the form meet acceptable presentation standards of scholarly work in the above mentioned discipline.

Brown-Dymkoski, Eric (M.S., Mechanical Engineering)

Computational Methodology for Aeroacoustic Simulation Using a Volume Penalization Method and  
CAD Generated Geometry

Thesis directed by Prof. Oleg V. Vasilyev

The problem of computational aeroacoustic modelling is complex and of practical interest, especially for external flows around arbitrary geometries. Aeroacoustic interactions are more sensitive to errors than aerodynamics, and as such, particular care must be taken to accurately and efficiently model them.

For efficiency, an adaptive multi-resolution grid is used to reduce the number of grid points while still resolving pertinent scales. The compressible Navier-Stokes equations are solved using the Adaptive Wavelet Collocation Method (AWCM), where wavelet decomposition provides a fast and efficient method for grid compression while maintaining rigorous control over the error.

The primary focus of this thesis is developing methodologies for efficient handling of solid obstacles within flows. Proper modelling of arbitrarily shaped obstacles is a prominent issue for fluid simulation, for which there are several approaches. Immersed boundary methods are well suited for use with rectilinear grids as they circumvent the need for a body-conformal mesh and allow curved geometry. The geometry can be efficiently generated through external CAD software, and ray-tracing algorithms used to create accurate masking functions. Ray-tracing is attractive for parallel computational systems as each grid point can be analyzed independently. A masking function provides the geometry definition for volume penalization methods.

A new volume penalization method is introduced here to address several difficulties associated with the Brinkman Penalization Method (BPM). Brinkman penalization is generally limited to a select group of fluid problems, and is greatly restricted in the handling and types of boundary conditions. To overcome these issues, a new, characteristic-based volume penalization method is proposed, allowing for general Dirichlet and Neumann boundary conditions to be defined.

## Dedication

To my grandfather, Howard Brown.

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## Chapter 1

### Introduction

#### 1.1 Motivation and Objective

The problem of flow-generated noise is of increasing engineering significance in the modern world. It was engine noise by jet aircraft that motivated Sir James Lighthill to derive the first aeroacoustic analogy in the middle of the 20th century [15]. Any situation that arises where there exists an unsteady fluid flow produces an acoustic field, and in some cases, this field can be quite significant. In aviation, automotive, and transportation industries, noise pollution adversely affects passenger comfort and is of concern to the surrounding communities. There are even aeroacoustic considerations for the energy industry. With wind turbines' increased popularity, noise generation impacts site selection and can retard site approval.

In some cases, the impact of flow noise goes beyond discomfort, and can affect technological performance. In the marine environment where acoustic sensing is paramount, such as sonar mapping or undersea warfare, anticipating and eliminating self-generated sound interference becomes a mission critical concern. Noise generated by a sensing platform at frequencies of interest is detrimental to performance. Furthermore, flow noise from propellers and hull/airframe protrusions adversely affects platform stealth for both aviation and marine platforms. Characterization and prediction of aeroacoustic spectra can aid in the mitigation process from the earliest design stages.

Many aeroacoustic problems of practical significance arise from flow past complex bodies. Therefore, there is a need to develop effective computational tools to accurately simulate arbitrary shapes. Difficulties manifest in generating realistic geometries and integrating them into a solver

environment. Computational meshing of applied geometry tends to contribute heavily to the computational cost of a numerical solution, and, additionally, particular care must be given to the quality of the flow model itself. Paramount to developing efficient models is minimizing the size of a computational domain, and therefore implementing nonreflecting boundaries. Because of a solution’s aeroacoustic sensitivity to spurious waves in pressure, it is particularly important that artificial computational boundaries do introduce significant nonphysical forcing into the domain.

The objective of this thesis is to develop an efficient simulation methodology for flow around complex geometries, with an emphasis on acoustic generation and interactions. While body-conformal meshes are straightforward in their treatment of solid obstacle boundary conditions, effective meshing can be quite costly. Immersed boundary methods eliminate the need for body fitting, and are even suitable for rendering curves on rectilinear grids. The Brinkman Penalization Method (BPM) has been shown to be quite effective for a wide range of fluid problems [3, 25], including computational aeroacoustics [2, 22]. Originally applied to incompressible flows, Brinkman penalization has been effectively extended to fully compressible, viscous regimes [21].

However, there are several drawbacks to BPM, including the lack of generality in defining Neumann boundary conditions. Furthermore, steep gradients can cause computational difficulties which have been previously handled through changes in the differencing stencil [2] and diffuse interfaces [25]. To overcome these difficulties, a new volume penalization method is proposed for the introduction of solid obstacles within the flow. This method provides general treatment for Neumann conditions, making the method suitable for a wide variety of fluid and non-fluid simulation.

## 1.2 Methodology

Computational aeroacoustics is fundamentally rooted in determining the time-accurate solution of a flow regime. Here, external compressible flow is considered around an obstacle of arbitrary shape. External flow is applicable to a wide variety of systems of engineering interest. One practical difficulty of such a problem lies in the scarcity of computational resources. External flows

often contain multiple temporal and spatial scales to be resolved. Efficient computation requires a grid that adapts automatically to resolve local scales and is able to follow the transient structures. Often, this can involve re-meshing steps that can be quite expensive.

The adaptive nature of wavelet modelling lends itself well to an efficient multi-resolution grid. Use of wavelet based adaptation has shown to be effective in a wide variety of fluid models [25, 26, 36], including compressible external flows [17]. The Adaptive Wavelet Collocation Method (AWCM) is used here to resolve multiple scales present in the solution.

Since AWCM utilizes a rectilinear grid, it precludes body-conformal grids for modelling solid obstacles. Conformal meshing is also undesirable in that the computational costs for effective grid generation are often quite high. Alternatively, immersed boundary methods are used here. A new characteristic-based volume penalization (CBVP) method is proposed here to model solid obstacles and overcome difficulties associated with Brinkman penalization. The error of CBVP can be rigorously controlled, making it appropriate for the high accuracies needed for aeroacoustic simulation.

For the prediction of flow generated noise, there are two principle methods of computational aeroacoustic modelling [37]. Direct methods calculate the sound concurrently with the flow field solution, accurately resolving acoustic scales on the computational mesh. This method has high computational costs associated with the acoustic scales, as well as requiring a domain that physically encompasses the radiation field of interest. Often, though, the sound-generating structures in the flow are limited to a region that is compact in comparison to the radiation field. In these cases, hybrid methods implementing aeroacoustic analogy, can be used to separate the calculation of aeroacoustic source structures with that of the far-field sound propagation. There, an accurate solution to the flow field only needs to encompass sound-generating structures and acoustic analogy determines the acoustic radiation. Hybrid methods assume a one-way decoupling of the acoustic and flow solutions, where the flow generates acoustic waves but these waves do not affect the flow. A popular aeroacoustic analogy for an external flow regime containing solid obstacles is the Ffowcs-Williams and Hawkings analogy which models radiation using linear acoustics. To circumvent

errors associated with finite time intervals, a method has been developed by Lockard to solve the Ffowcs-Williams and Hawkings equation in the frequency domain [23].

The acoustic solutions themselves are more sensitive than aerodynamic solutions. As such, aeroacoustic methodologies are particularly sensitive to spurious waves. The treatment of boundary conditions, especially in external flow, can be a major cause of nonphysical reflections. Fluid modelling constantly pursues transparent boundaries that allow flow structures to exit the domain unimpeded. Freund has developed a zonal method for nonreflecting boundaries that introduces artificial damping and convective terms in a narrow zone around the domain edges [11]. The strength of the method lies in its computationally simple approach for maintaining accuracy in the physical domain.

### 1.3 Organization

The rest of this thesis is organized as follows. Chapter 2 outlines the necessary background for aeroacoustics and the methods used along with the dominant physics and constitutive equations. The computational methodologies used for nonreflective boundary conditions, solid obstacle modelling, and aeroacoustic analysis are described. Chapter 3 discusses the extension of automated CAD geometry analysis and aeroacoustic analogy for parallel computing. Finally, a new characteristic-based volume penalization method is introduced in Chapter 4. This method is inspired by the Brinkman penalization method, but allows for greater control over specific immersed boundary conditions. In this chapter, a series of single- and multi-dimensional benchmark problems are used to validate the new penalization method and the aeroacoustic methodologies. The final chapter outlines conclusions and comments upon future research on this topic.



## Chapter 2

### Background

#### 2.1 Aeroacoustics

##### 2.1.1 Classical Acoustics

In linear acoustic theory, acoustic waves are considered solutions of the inhomogeneous wave equation, in pressure,

$$\left( \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) p = \mathcal{F}(\mathbf{x}, t), \quad (2.1)$$

where  $\mathcal{F}(\mathbf{x}, t)$  is any generalized source, or collection of sources, in space and time. Solutions of the wave equation without the presence of solid boundaries are built upon the free space Green's function

$$G(\mathbf{x}, \mathbf{y}, t - \tau) = \frac{1}{4\pi|\mathbf{x} - \mathbf{y}|} \delta \left( t - \tau - \frac{|\mathbf{x} - \mathbf{y}|}{c_0} \right). \quad (2.2)$$

In three dimensions, this is an expanding sphere centered at  $\mathbf{y}$ , and zero for any time  $t < \tau$ . The Green's function is impulsive and nonzero only at the distance  $r = |\mathbf{x} - \mathbf{y}|$  from the center  $\mathbf{y}$ . As a solution to (2.1), it is the wave, propagating outward at velocity  $c_0$ , instigated by an impulsive point source  $\delta(\mathbf{x} - \mathbf{y}) \delta(t - \tau)$  located at  $\mathbf{x} = \mathbf{y}$ , and triggered at time  $t = \tau$ . The differential form of Green's function as an inhomogeneous wave is

$$\left( \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) G = \delta(\mathbf{x} - \mathbf{y}) \delta(t - \tau). \quad (2.3)$$

If we consider that any general source term  $\mathcal{F}(\mathbf{x}, t)$  of (2.1) can be taken as the sum of a collection of impulsive point sources,

$$\mathcal{F}(\mathbf{x}, t) = \int \int_{-\infty}^{\infty} \mathcal{F}(\mathbf{y}, \tau) \delta(\mathbf{x} - \mathbf{y}) \delta(t - \tau) d^3\mathbf{y} d\tau, \quad (2.4)$$

then the Green's function can be used to construct further solutions of (2.1). In this way, the acoustic radiation is the linear superposition of impulsive point disturbances in pressure, given as

$$\begin{aligned} p(\mathbf{x}, t) &= \int \int_{-\infty}^{\infty} \mathcal{F}(\mathbf{y}, \tau) G(\mathbf{x}, \mathbf{y}, t - \tau) d^3\mathbf{y} d\tau \\ &= \frac{1}{4\pi} \int \int_{-\infty}^{\infty} \frac{\mathcal{F}(\mathbf{y}, \tau)}{|\mathbf{x} - \mathbf{y}|} \delta\left(t - \tau - \frac{|\mathbf{x} - \mathbf{y}|}{c_0}\right) d^3\mathbf{y} d\tau \\ &= \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{\mathcal{F}\left(\mathbf{y}, t - \frac{|\mathbf{x} - \mathbf{y}|}{c_0}\right)}{|\mathbf{x} - \mathbf{y}|} d^3\mathbf{y}. \end{aligned} \quad (2.5)$$

Acoustic waves are generally thought of as deviations of variables from the mean flow, especially considering the impulsive nature of Green's function. For this reason it is useful to introduce perturbation quantities as a more intuitive form of acoustic variables, where

$$\begin{aligned} \rho' &= \rho - \rho_0 \\ p' &= p - p_0. \end{aligned}$$

Since the undisturbed state  $(p_0, \rho_0)$  is typically constant, solutions to the wave equation are the nonzero perturbations propagating through the medium.

In classical acoustics, the generation of acoustic waves in fluids is attributed to a few distinct phenomena, or sources. These phenomena are, individually and collectively, source terms  $\mathcal{F}(\mathbf{x}, t)$  in the linear wave equation (2.1) [19]. The first of these is the simple, or monopole, source consisting of a time varying fluctuation of mass at a point,  $q(t)\delta(\mathbf{x})$ . For a monopole source at the origin, the forcing of the wave equation is the unsteady mass flux

$$\mathcal{F}(\mathbf{x}, t) = -q(t)\delta(\mathbf{x}). \quad (2.6)$$

The radiated pressure waves take on the form

$$p'(\mathbf{x}, t) = -\frac{q\left(t - \frac{|\mathbf{x}|}{c_0}\right)}{4\pi|\mathbf{x}|}, \quad (2.7)$$

and

$$p'(\mathbf{x}, t) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{q\left(\mathbf{y}, t - \frac{|\mathbf{x}-\mathbf{y}|}{c_0}\right)}{|\mathbf{x} - \mathbf{y}|} d^3\mathbf{y}, \quad (2.8)$$

for a point and distributed source, respectively. The radiation of acoustic waves from a simple source is nondirectional in nature.

The second source can be thought of as the summation of two monopole sources an infinitesimal distance apart. If the strengths of these sources are equal and opposite, the net mass flux is zero and the total monopole strength is zero. However, the wave equation is still satisfied, with a nonzero source term taking on the form of an unsteady force acting upon the fluid. A dipole source consisting of a volume force acting in the  $j$ th direction at the origin,

$$\mathcal{F} = \nabla \cdot (\mathbf{F}(t)\delta(\mathbf{x})) = \frac{\partial F_j(t)\delta(\mathbf{x})}{\partial x_j}, \quad (2.9)$$

will create acoustic pressure perturbations

$$p'(\mathbf{x}, t) = \frac{\partial}{\partial x_j} \left( \frac{F_j\left(t - \frac{|\mathbf{x}|}{c_0}\right)}{4\pi|\mathbf{x}|} \right). \quad (2.10)$$

The solution for a distribution of dipoles is

$$p'(\mathbf{x}, t) = \frac{1}{4\pi} \frac{\partial}{\partial x_j} \int_{-\infty}^{\infty} \frac{F_j\left(\mathbf{y}, t - \frac{|\mathbf{x}-\mathbf{y}|}{c_0}\right)}{|\mathbf{x} - \mathbf{y}|} d^3\mathbf{y}. \quad (2.11)$$

The radiation field of the dipole source is a bi-directional figure eight pattern in the direction of the acting force. Considering the dipole as the sum of two infinitesimally close, opposing monopoles, it is rightly expected that the strength of a dipole is significantly less than the sum of the two individual monopoles. The dipole is a less efficient radiator of energy. Dipoles exist whenever fluctuating forces exist in the fluid, and are often marked by the presence of solid objects. Vibration of machinery or structures creates dipole acoustic radiation.

The third fundamental source can be thought of as the result of two opposing dipoles superimposed an infinitesimal distance apart. The net force at any given instant is zero. However, the effect of the nonzero stresses from the dipole summation will still provide forcing for a solution to

the wave equation. The quadrupole source is the volumetric stress

$$\mathcal{F}(\mathbf{x}, t) = \frac{\partial^2 T_{ij}(\mathbf{x}, t)}{\partial x_i \partial x_j}. \quad (2.12)$$

The pressure fluctuations for a point and distributed quadrupole are

$$p'(\mathbf{x}, t) = \frac{\partial^2}{\partial x_i \partial x_j} \frac{T_{ij} \left( t - \frac{|\mathbf{x}|}{c_0} \right)}{4\pi |\mathbf{x}|} \quad (2.13)$$

and

$$p'(\mathbf{x}, t) = \frac{1}{4\pi} \frac{\partial^2}{\partial x_i \partial x_j} \int_{-\infty}^{\infty} \frac{T_{ij} \left( \mathbf{y}, t - \frac{|\mathbf{x}-\mathbf{y}|}{c_0} \right)}{|\mathbf{x} - \mathbf{y}|} d^3 \mathbf{y}. \quad (2.14)$$

Quadrupoles radiate in a cloverleaf pattern, and their strength is less than the combined strength of the constitutive dipoles, for the same reasons as before. The efficiency of a quadrupole acoustic source is less than that of the monopole or dipole.

Since higher order sources are still less efficient, discussion typically stops at the quadrupole. Many aeroacoustic analogies, including the Ffowcs Williams and Hawkings equation, only consider the effects of monopole, dipole, and quadrupole sources.

### 2.1.2 Aeroacoustic Analogy

Aeroacoustic analogies are based upon reformulation of the constitutive flow equations,  $\mathcal{N}(\mathbf{q}) = 0$ , into a form analogous of a wave equation. The general form of aeroacoustic analogies follows

$$\mathcal{L}(\mathbf{q}) = \mathcal{S}(\mathbf{q}),$$

where  $\mathbf{q}$  is the set of variables that comprise the solution of the flow equations,  $\mathcal{L}$  is a wave operator, and  $\mathcal{S}$  is analogous to the corresponding source terms [37]. The various aeroacoustic analogies differ in the form of the wave operator  $\mathcal{L}$ . The obvious shortcoming, as pointed out by multiple authors [15, 37] is that the wave operator  $\mathcal{L}$  and the sources  $\mathcal{S}$  are both functions of the flow variables  $\mathbf{q}$ . The propagation is part of the same flow regime from which it is sourced. In this respect, it is not precisely a wave equation, but rather only analogy. A principle assumption with aeroacoustic analogy is that acoustic energies are relatively low, and there is a one-way decoupling of the acoustics

from the flow. The flow field generates the acoustic perturbations, but the perturbations themselves do not affect the flow [15].

The most famous, and historically earliest, of the aeroacoustic analogies was derived by Lighthill in 1952. Specifically motivated by the problem of jet noise, it was built around a steady free jet in a quiescent fluid [18]. In Lighthill's analogy, continuity and momentum equations are rearranged into the form of the linear wave equation (2.1):

$$\left( \frac{\partial^2}{\partial t^2} - c_0^2 \nabla^2 \right) \rho = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}, \quad (2.15)$$

where

$$T_{ij} = \rho u_i u_j + (p - c_0^2 \rho) \delta_{ij} - \sigma_{ij}. \quad (2.16)$$

Noting that  $c_0^2 = \partial p / \partial \rho$  and  $p' = p - p_0$ , the more familiar form of the acoustic wave equation emerges:

$$\left( \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) p' = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}. \quad (2.17)$$

In this form, Lighthill has assumed that there are no solid obstacles in the flow and that any mass sources/sinks are steady state. The result is that his aeroacoustic analogy describes unbounded flow with a quiescent state at infinity, where the stress tensor  $T_{ij}$  describes a quadrupole source. The form of the solution is straightforward, as shown in equation (2.14), however complete knowledge of the flow is needed to define the source,  $T_{ij}$ .

Williams and Hawkings extended this analogy to include the presence of surfaces in the flow. With an arbitrary surface, monopole and dipole sources emerge, accounting for the possible presence of solid obstacles.

Solving aeroacoustic analogies requires time-accurate knowledge of the flow regime. Because of the difficulty in obtaining analytical solutions to problems of engineering interest, accurate computational solutions of the flow field are often well suited for prediction of aeroacoustic noise.

## 2.2 Governing Equations

The problem considered is external flow of a compressible, viscous fluid on  $\mathbb{R}^2$  or  $\mathbb{R}^3$  in two or three dimensions. The solution is found in Cartesian coordinates on a domain,  $\Omega = [L_1, L_2] \times [M_1, M_2]$ , containing all obstacles  $O_i$ . A hybrid approach to predicting the flow-generated sound is used, where the aeroacoustic solution is calculated after the transient flow has been determined. Direct numeric simulation (DNS) of the constitutive equations is used for the low Reynolds number flows considered here.

### 2.2.1 Conservation Laws

The fluid flow is governed by the compressible Navier-stokes equations. For a domain with position vector  $\mathbf{x}$  and fluid velocity  $\mathbf{u}$ , the continuity, momentum and energy equations in conservative form are

$$\frac{\partial \rho}{\partial t} = -\frac{\partial \rho u_j}{\partial x_j}, \quad (2.18)$$

$$\frac{\partial \rho u_i}{\partial t} = -\frac{\partial (\rho u_i u_j)}{\partial x_j} - \frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j}, \quad (2.19)$$

$$\frac{\partial \rho e}{\partial t} = -\frac{\partial}{\partial x_j} [(\rho e + p) u_j] + \frac{\partial (u_i \tau_{ij})}{\partial x_j} + \frac{\partial}{\partial x_j} \left( k \frac{\partial T}{\partial x_j} \right), \quad (2.20)$$

where  $\rho$  is the density of the fluid,  $\rho u_i$  is the mass flux,  $p$  is the pressure, and  $k$  is the thermal conductivity. The shear stress tensor is

$$\tau_{ij} = \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \frac{\partial u_k}{\partial x_k} \delta_{ij} \right),$$

and the total specific energy is

$$e = \frac{1}{2} u_i u_i + c_p T - \frac{p}{\rho},$$

where  $p = \rho R T$ , the gas constant is  $R = c_p(\gamma - 1)/\gamma$ , the specific heat ratio is  $\gamma = \frac{c_p}{c_v}$ , and  $\mu$  is the dynamic viscosity.

For the computational model, the nondimensionalized Navier-Stokes equations corresponding

to (2.18), (2.19) and (2.20) are

$$\frac{\partial \rho^*}{\partial t^*} = -\frac{\partial \rho^* u_j^*}{\partial x_j^*}, \quad (2.21)$$

$$\frac{\partial \rho^* u_i^*}{\partial t^*} = -\frac{\partial(\rho^* u_i^* u_j^*)}{\partial x_j^*} - \frac{\partial p^*}{\partial x_i^*} + \frac{1}{Re_a} \frac{\partial \tau_{ij}^*}{\partial x_j^*}, \quad (2.22)$$

$$\frac{\partial \rho^* e^*}{\partial t^*} = -\frac{\partial}{\partial x_j^*} [(\rho^* e^* + p^*) u_j^*] + \frac{1}{Re_a} \frac{\partial(u_i^* \tau_{ij}^*)}{\partial x_j^*} + \frac{1}{(\gamma - 1)} \frac{1}{Re_a Pr} \frac{\partial}{\partial x_j^*} \left( \mu \frac{\partial T^*}{\partial x_j^*} \right), \quad (2.23)$$

where

$$\begin{aligned} p^* &= \frac{\rho^* T^*}{\gamma}, \\ \tau_{ij}^* &= \mu^* \left( \frac{\partial u_i^*}{\partial x_j^*} + \frac{\partial u_j^*}{\partial x_i^*} - \frac{2}{3} \frac{\partial u_k^*}{\partial x_k^*} \delta_{ij} \right), \\ e^* &= \frac{1}{2} u_i^* u_i^* + c_p T^* - \frac{p^*}{\rho^*}, \end{aligned}$$

$Re_a$  is the acoustic Reynolds number and  $Pr$  is the Prandtl number. Length scales are based on the characteristic length of the obstacle,  $L$ . The velocity  $\mathbf{u}$  is nondimensionalized based on the reference speed of sound  $c_0$ , time based on  $L/c_0$ , specific energy on  $c_0^2$ , density on  $\rho_0$ , pressure on  $\rho_0 c_0^2$ , viscosity on  $\mu_0$ , thermal conductivity on  $\mu_0 c_{p0}$ , and temperature on  $T_0$ . Note that nondimensional pressure  $p^*$  is not based on a reference pressure  $p_0$ .

### 2.2.2 Ffowcs Williams and Hawkings Method

In this nondirect approach, the acoustic scales and radiation are not resolved for concurrently with the flow solution. Instead, the sound radiation is treated via acoustic analogy. The Ffowcs Williams and Hawkings equation is solved across the computational domain using the time accurate solution as the source.

The Ffowcs Williams and Hawkings (FWH) aeroacoustic analogy applies to flow regimes in  $\mathbb{R}^3$  or  $\mathbb{R}^3$  containing a rigid closed surface  $S$ . A volume function  $f$  is prescribed by the surface, where

$$f(t) = \begin{cases} f > 0 & \text{outside } S \\ f < 0 & \text{inside } S \end{cases}$$

Likewise, the Heaviside function is described by  $H(f) = 1$  for  $f > 0$  and  $H(f) = 0$  for  $f < 0$ . The spatial derivative,  $\partial f / \partial x_i$  is nonzero only upon the surface,  $S$  [15].

Under these conditions, and considering fluxes in flow variables across  $S$ , the continuity and momentum equations can exactly be rearranged into the differential forms

$$\left( \frac{\partial^2}{\partial t^2} - c_0^2 \nabla^2 \right) (H(f)\rho) = \frac{\partial^2 (T_{ij}H(f))}{\partial x_i \partial x_j} - \frac{\partial (F_i \delta(f))}{\partial x_i} + \frac{\partial (Q \delta(f))}{\partial t}, \quad (2.24)$$

where  $T_{ij}$  again is the Lighthill stress tensor (2.16). In comparing the source terms of equation (2.24) with the fundamental acoustic sources, (2.6) (2.9), we find that the dipole term is

$$F_i = (P_{ij} + \rho u_i (u_j - v_j)) \frac{\partial f}{\partial x_i}$$

and a monopole is given by

$$Q_i = (p_0 v_i + \rho (u_i - v_i)) \frac{\partial f}{\partial x_i},$$

where  $\mathbf{v}$  is the velocity of the surface  $S$  and  $P_{ij}$  is the compressive stress tensor. As in Lighthill's analogy (2.17), we see that  $T_{ij}H(f)$  describes a quadrupole outside of the surface  $S$ . Additionally,  $F_i$  describes a dipole on  $S$  and  $Q_i$  a monopole on  $S$ . Using the Green's function, the retarded potential solution of the Ffowcs Williams and Hawkings equation, given in acoustic perturbation quantities, is

$$\begin{aligned} p'(\mathbf{x}, t) = & \frac{1}{4\pi} \frac{\partial^2}{\partial x_i \partial x_j} \int_V \frac{T_{ij}(\frac{|\mathbf{x}-\mathbf{y}|}{c_0})}{|\mathbf{x}-\mathbf{y}|} d^3 \mathbf{y} \\ & - \frac{1}{4\pi} \frac{\partial}{\partial x_i} \int_S \frac{F_i(\frac{|\mathbf{x}-\mathbf{y}|}{c_0})}{|\mathbf{x}-\mathbf{y}|} d^2 \mathbf{y} \\ & + \frac{1}{4\pi} \frac{\partial}{\partial t} \int_S \frac{Q_i(\frac{|\mathbf{x}-\mathbf{y}|}{c_0})}{|\mathbf{x}-\mathbf{y}|} d^2 \mathbf{y}. \end{aligned} \quad (2.25)$$

In two dimensions, the solution (2.25) simply reflects the use of the appropriate two dimensional Green's function [23]. The monopole and dipole sources on the surface  $S$  are the result of the changing volume enveloped by  $S$  and the unsteady forces present along the boundary. The integral across the stress tensor  $T_{ij}$  considers the perturbations caused by shear stresses in the volume outside of the surface. It is important to note that shear stresses within  $S$  manifest as lower order sources on  $S$  [23].



In many practical cases, the FWH equation can be simplified by assuming that the surface is stationary and constant volume in  $\mathbf{x}$ . Under these assumptions, the monopole source in (2.25) vanishes, and the dipole source  $F_i$  is greatly simplified. In this form, the FWH analogy is known as Curle's equation [15].

Typically, we are concerned with noise that propagates large distances where the observer is located a distance that is many wavelengths away from the source region. Further assuming that the source region is compact, where the distance from the observer  $|\mathbf{x} - \mathbf{y}|$  is large compared to the length scale of the source region, Curle's equation for the acoustic far-field becomes

$$p'(\mathbf{x}, t) \approx \frac{1}{4\pi} \frac{\partial^2}{\partial x_i \partial x_j} \int_V \frac{T_{ij}(\frac{|\mathbf{x}-\mathbf{y}|}{c_0})}{|\mathbf{x}|} d^3\mathbf{y} - \frac{1}{4\pi} \frac{\partial}{\partial x_i} \int_S \frac{F_i(\frac{|\mathbf{x}|}{c_0})}{|\mathbf{x}-\mathbf{y}|} d^2\mathbf{y}. \quad (2.26)$$

## 2.3 Computational Methodologies

### 2.3.1 Adaptive Wavelet Collocation

Aeroacoustic problems, and many general fluid problems, contain many disparate temporal and spatial scales. For such problems, discretizing constitutive equations on a constant resolution mesh is inefficient. High resolutions are only needed at locations where sharp transitions occur. In regions with gradual spatial variation, fewer points are needed to adequately resolve the solution. Furthermore, disparate structures are often transient, requiring a continuously adaptive grid to resolve dissimilar scales.

The adaptive wavelet collocation method (AWCM) is a technique for dynamically adapting a rectilinear mesh to local scales [32, 33, 34, 35]. A solution,  $u(\mathbf{x})$ , on a discrete grid is approximated using a set of wavelets, where there is a one-to-one correspondence between grid points and wavelets. Filtering out insignificant wavelets from the computational domain can greatly compress the number of mesh points, while still rigorously maintaining desired accuracy.

Second generation wavelets are used in AWCM, where translational/dilational relationships between wavelets and a mother wavelet have been abandoned [34]. First, considering an array of

points  $\mathbf{x}$  on a closed interval  $\Omega$  that form a set of  $\mathcal{J}$  nested grids,

$$\mathcal{G}^j = \{x_k^j \in \Omega : k \in \mathcal{K}^j\}, j \in \mathcal{J}. \quad (2.27)$$

By enforcing  $x_k^j = x_{2k}^{j+1}$ , the nestedness of the grids,  $\mathcal{G}^j \subset \mathcal{G}^{j+1}$ , is ensured. By desired scaling functions  $\phi_k^j(\mathbf{x})$  and wavelets  $\psi_l^j(\mathbf{x})$ , a function  $u(\mathbf{x})$  can be decomposed [34] by

$$u(\mathbf{x}) = \sum_{k \in \mathcal{K}^0} c_k^0 \phi_k^0(\mathbf{x}) + \sum_{j=0} \sum_l d_l^j \psi_l^j(\mathbf{x}). \quad (2.28)$$

By truncating the summation over the  $j_{\text{th}}$  index at some upper resolution level  $\mathcal{J}$ , the function  $u(\mathbf{x})$  is approximated by  $u^{\mathcal{J}}(\mathbf{x})$ . The wavelet coefficients,  $d_l^j$ , are representative of that wavelet's contribution to the solution. Eliminating wavelets whose coefficients fall below some *a priori* threshold  $\epsilon$  allows for a large number of points to be disregarded, while only incurring an error of  $O(\epsilon)$ . The remaining set comprises the significant wavelets, defined as those having a coefficient above the threshold value, that is  $|d_l^j| \geq \epsilon$ .

To examine the error incurred in the wavelet approximation, we group the wavelets in equation (2.28) into those above and below the threshold  $\epsilon$ ,

$$u(\mathbf{x}) = u_{\geq}(\mathbf{x}) + u_{<}(\mathbf{x}). \quad (2.29)$$

From the wavelet transform, (2.28), the wavelet groups are

$$u_{\geq}(\mathbf{x}) = \sum_{k \in \mathcal{K}^0} c_k^0 \phi_k^0(\mathbf{x}) + \sum_{j=0} \sum_{\substack{l \\ |d_l^j| \geq \epsilon}} d_l^j \psi_l^j(\mathbf{x}), \quad (2.30)$$

$$u_{<}(\mathbf{x}) = \sum_{j=0} \sum_{\substack{l \\ |d_l^j| < \epsilon}} d_l^j \psi_l^j(\mathbf{x}). \quad (2.31)$$

From these relationships, it can be shown [9, 33, 34] that

$$|u(\mathbf{x}) - u_{\geq}(\mathbf{x})| \leq C_1 \epsilon \leq C_2 N^{-p/n}, \quad (2.32)$$

where  $N$  is the number of significant wavelets,  $p$  is the order of the wavelets, and  $n$  is the dimensionality of the problem. The coefficients  $C_1$  and  $C_2$  are dependent upon the function  $u^{\mathcal{J}}(\mathbf{x})$ , but are  $O(1)$ .

To realize domain compression and adaptation, we acknowledge that each grid point  $\mathbf{x}$  uniquely corresponds to a collocation point for a wavelet. Since  $u(\mathbf{x})$  can be approximated by  $u_{\geq}(\mathbf{x})$  on an  $N$  number of wavelets, all non-significant wavelets and their corresponding grid points can be discarded. The result is a nested set of computational grids  $\mathcal{G}_{\geq}^{\mathcal{J}} \subset \mathcal{G}^{\mathcal{J}}$  where  $\mathcal{G}^{j+1} \subset \mathcal{G}^j$  with areas of high resolution clustered about regions of sharp transition in  $u(\mathbf{x})$ . The nature of the adaptive grid  $\mathcal{G}_{\geq}^{\mathcal{J}}$  is highly dependent upon  $u(\mathbf{x})$ .

For transient problems, the grid  $\mathcal{G}_{\geq}^{\mathcal{J}}(t)$  is time dependent. As the solution evolves, regions of sharp local structures may change position in  $\Omega$ , and in order to maintain accuracy, changes in  $u(\mathbf{x})$  are anticipated by adding the nearest neighbouring points of significant wavelets to the grid  $\mathcal{G}_{\geq}^{\mathcal{J}}$  [32].

Based upon the compression and interpolation properties of wavelets, the numerical algorithm for adapting the grid to a transient solution proceeds in a straightforward fashion [34]:

- (1) The solution  $u^{\mathcal{J}}(t)$  is known from a previous timestep or the initial conditions, and exists on  $\mathcal{G}_{\geq}^{\mathcal{J},t}$  for time  $t$ . Based on the magnitude of the wavelet coefficients in (2.28) and a given  $\epsilon$ , the grid  $\mathcal{G}_{\geq}^{\mathcal{J},t+\Delta t}$  is created to include nearest neighbours of  $\mathbf{x}_i^j$  for all  $j > j^{\min}$ .
- (2) If  $\mathcal{G}_{\geq}^{\mathcal{J},t}$  and  $\mathcal{G}_{\geq}^{\mathcal{J},t+\Delta t}$  are the same, proceed directly to step 3. Otherwise, the solution is interpolated for the new points in  $\mathcal{G}_{\geq}^{\mathcal{J},t+\Delta t}$ .
- (3) Integrate the differential system to obtain the new solution  $u_{\geq}^{\mathcal{J}}(t + \Delta t)$  on  $\mathcal{G}_{\geq}^{\mathcal{J},t+\Delta t}$ , and go back to step 1.

Since a dyadic, nested grid is used, adaptation from  $j$  to  $j + 1$  will resolve features at half the scale. With this adaptation scheme, AWCM not only achieves high compression of the computational grid, but the  $L^{\infty}$ -norm error is strictly controlled. By prescribing a lower threshold,  $\epsilon$ , and a sufficiently high  $\mathcal{J}$  to capture all significant wavelets, the error can be systematically reduced by including a higher number of wavelets. The adaptation algorithm itself, however, is computationally cheap and can be effectively implemented at every timestep.

### 2.3.2 Brinkman Penalization

Introduction of solid objects within the domain requires treatment of the domain and constitutive equations in the edges and body of the obstacle. Specified boundary conditions must be rigorously enforced along surfaces. Where body-fitting meshes are used, prescription of conditions proceeds in a natural fashion. However, meshing of arbitrarily complex geometry can be expensive, especially in an adaptive environment. Re-meshing and identifying surface nodes can comprise a large portion of the solution computational costs.

Immersed boundary methods have been developed to simulate the flow around arbitrary complex geometry on a non-body conformal grid. The Brinkman penalization method achieves this by penalizing the flow equations within the obstacle and modelling it as porous media. Porous media consists of a solid matrix with interconnected pores allowing fluid penetration. The volume fraction of pores in the media is the porosity  $\phi$ . For a porosity  $\phi \ll 1$ , the media could be considered to resemble a solid obstacle.

The continuity equation for porous media follows naturally [21],

$$\frac{\partial \rho}{\partial t} = -\frac{1}{\phi} \frac{\partial \rho u_j}{\partial x_j}. \quad (2.33)$$

The momentum equation was first derived for incompressible flows, and later extended to the compressible regime. For porous media, Darcy's law for fluid momentum balances the fluid pressure  $p$  with viscous stresses [6],

$$\mathbf{v} = -\frac{K}{\mu} \nabla p, \quad (2.34)$$

where  $\mathbf{v}$  is the seepage velocity given by the Dupuit-Forchheimer relationship  $\mathbf{v} = \phi \mathbf{u}$ . Addition of a viscous term, Darcy's law is extended into the Brinkman equation [3],

$$\nabla p = -\frac{\mu}{K} \mathbf{v} + \mu \nabla^2 \mathbf{v}. \quad (2.35)$$

Combining the Brinkman and continuity equations into a form analogous of the incompressible Navier-Stokes equations [31] yields a conservative form of the momentum equation for fluid in a

porous media,

$$\frac{1}{\phi} \frac{\partial \rho v_i}{\partial t} = -\frac{1}{\phi} \frac{\partial}{\partial x_j} (\rho \phi^{-1} v_i v_j) - \frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 v_i}{\partial x_j^2} - \frac{\mu}{K} v_i \quad (2.36)$$

where  $\phi = \phi(\mathbf{x})$ . Location dependent porosity is considered in order to model porous media flow in all of the domain,  $\Omega$ , as obstacles within an open flow regime. For the region external to obstacles,  $\phi$  is set to 1, and internal to the obstacles,  $\phi \ll 1$ . Liu notes several instances where the momentum equation (2.36) can be greatly simplified for use in numerical simulation [21].

Lastly, the energy equation for flow through porous media is simply the compressible energy equation (2.20) with a forcing term to maintain a specified temperature,

$$\frac{\partial \rho e}{\partial t} = -\frac{\partial}{\partial x_j} [(\rho e + p) v_j] + \frac{\partial}{\partial x_j} \left( k \frac{\partial T}{\partial x_j} \right) - \frac{h}{\phi} (T - T_0). \quad (2.37)$$

For simplicity, the heat transfer coefficient  $h$  is included in a normalized thermal permeability  $\eta_T = \phi/h$ .

In using these equations to adequately simulate obstacles in external flow, no-slip boundary conditions need to be strictly enforced for obstacles  $O_i$ ,

$$\mathbf{u} = \mathbf{U}_0 \text{ on } \partial O_i, \quad (2.38)$$

where  $\mathbf{U}_0$  is the obstacle velocity. Angot et al. [1] model this effect for incompressible flows by using Brinkman penalization with the  $L^2$ -penalized equation

$$\frac{\partial u_i}{\partial t} = -\frac{\partial (u_i u_j)}{\partial x_j} - \frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j^2} - \frac{\chi}{\eta} (u_i - U_{0i}), \quad (2.39)$$

where

$$P = \frac{p}{\rho}. \quad (2.40)$$

Permeability,  $\eta = \alpha\phi$ , is a constant normalized viscous permeability. A masking function,  $\chi$ , is defined as

$$\chi = \begin{cases} 1 & \text{in } \mathbf{x} \in O_i, \\ 0 & \text{otherwise.} \end{cases}$$

Effectively,  $\chi$  provides support for the permeability  $\phi(\mathbf{x}) = \chi\phi$ , imposing  $\phi = 1$  outside of the obstacles.

Appropriate extension of Brinkman penalization to compressible regimes was achieved by Liu and Vasilyev [21]. In order to enforce no-slip conditions and a constant temperature obstacle, the compressible Navier-Stokes equations become

$$\frac{\partial \rho}{\partial t} = - \left[ 1 + \left( \frac{1}{\phi} - 1 \right) \chi \right] \frac{\partial \rho u_j}{\partial x_j}, \quad (2.41)$$

$$\frac{\partial \rho u_i}{\partial t} = - \frac{\partial (\rho u_i u_j)}{\partial x_j} - \frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} \left[ - \frac{\chi}{\eta} (u_i - U_{0i}) \right], \quad (2.42)$$

$$\frac{\partial \rho e}{\partial t} = - \frac{\partial}{\partial x_j} [(\rho e + p) u_j] + \frac{\partial (u_i \tau_{ij})}{\partial x_j} + \frac{\partial}{\partial x_j} \left( k \frac{\partial T}{\partial x_j} \right) \left[ - \frac{\chi}{\eta T} (T - T_0) \right], \quad (2.43)$$

For a solid obstacle within a flow regime, equations (2.41), (2.42) and (2.43) are solved everywhere. More notably, the treatment of derivatives can be uniform across the entire domain. Efficient finite differencing schemes can be employed, providing a marked advantage over body-conformal meshes [1].

There are a few drawbacks associated with Brinkman penalization. The introduction of low porosity on the continuity equation can lead to steep gradients, requiring modifications of the differencing stencil within the fluid region [2] or the use of diffuse-edge masking functions [25]. Furthermore, the BPM is limited in the application of Dirichlet boundary conditions. The use of porosity  $\phi$  mimics a no-flux boundary condition, but does not allow for application of general Neumann conditions. This makes Brinkman penalization in its current form unsuitable for fluid simulations involving specified heat fluxes.

### 2.3.3 Generation and Incorporation of Complex Geometry

The use of penalization methods depends on a description of a mask function,  $\chi$ , that accurately represents any arbitrary obstacle  $O_i$ . For simple geometry, this is readily accomplished through analytical functions to describe surfaces. However, for problems of engineering interest, the complexity of the geometry can make it impractical or impossible to create a satisfactory masking function analytically, especially in three dimensions.

The use of CAD software to generate desired geometry is a practical recourse, greatly simplifying obstacle definition. Determining  $\chi$  for a given discrete grid is the primary challenge to using

CAD models. For a set of points  $\mathbf{P}$  in  $\mathbb{R}$  containing a closed surface  $S$ , it must be found whether the points are located inside ( $\chi = 1$ ) or outside ( $\chi = 0$ ) of  $S$ . Automated analysis of points constrained by an arbitrary surface is a problem well known in computer graphics as well as in computational modelling. For in/out determination, consider a ray originating from point  $P_1$  and extending to infinity. If the ray intersects  $S$  an even number of times, it is located outside, if it intersects  $S$  an odd number of times, it is inside. If there are zero intersections, the point is definitively outside. In theory, the direction of the ray can be arbitrary, and only one ray is needed. Ghosh argues that this principle, known as the odd-parity rule, has a basis in Gauss' Law and electromagnetic field theory [14].

Analyzing CAD geometry requires a file format that defines solid body surfaces, as opposed to volumetric representation. Consider then, a closed surface  $S$  comprised of a piecewise continuous assembly of bounded analytical surfaces  $s\{s_i\}$ . Determination of point location is then based upon ray intersections with any  $s_i \subset S$ . Knowing the bounded points  $\xi|_{s_i}$  that form the surface  $s_i$ , determination of intersections from an arbitrary ray follows naturally. In developing an algorithm, Schneider and Eberly suggest a parametric ray of the form

$$\mathbf{m} = \mathbf{P}(\mathbf{x}) + \tau \hat{\mathbf{m}},$$

where  $\hat{\mathbf{m}}$  is the direction of ray [27]. If  $\xi = f(\mathbf{x})$ , and a  $\tau$  can be found so that

$$\xi|_{s_i} = \mathbf{P}(\mathbf{x}) + \tau \hat{\mathbf{m}}, \quad (2.44)$$

for a specified  $\hat{\mathbf{m}}$ , then intersection is determined for  $\mathbf{m}$  with  $s_i$ . Particular care must be taken for rays that intersect nodes and edges of  $s_i$ , and experience shows that multiple rays are needed to ensure robust analysis.

Some standardized CAD formats include representation of geometric surfaces by using a mesh of complex polygons. For non-planar surfaces, fine resolution with a large number of polygons is used. Odd-parity algorithms are relatively simple for such a format, as  $\xi|_{s_i}$  is planar and intersection easy to determine. As such, using poly-mesh representation is a popular method of computational analysis.

For universal representation of smooth contoured geometry, Initial Graphics Exchange Specification (IGES) files format is used. Aside from planar surfaces, IGES utilizes a wide variety of circular, revolved, and B-spline curves to accurately represent curved surfaces. The principle advantage of this is that fewer surfaces are needed to represent complex geometry at a greater precision. This is advantageous in fluid modelling to guarantee sufficient resolution of the geometry. In using IGES format, the resolution of a surface is limited only by the resolution of the mesh, which can be defined or changed based on the needs of the simulation without requiring file regeneration.

A standard file format, IGES is not proprietary to modelling platforms. This format is ASCII based, or compressed version thereof, and contains descriptions of five classes of entities: curve and surface geometries, constructive solids, boundary represented solids, annotation entities, and structures [30]. The format is maintained as a specification by the US Product Data Association, and as such, is readily available and accessible for use in the computational environment.

#### **2.3.4 Nonreflective Boundary Conditions**

In addition to the immersed boundaries, the domain boundaries must also be well posed and representative of the physics of the external flow. Unlike internal flows, where physical boundaries readily define the domain, external flows are typically modelled as being unbounded where the edges are transparent to the flow. Additionally, any interactions of fluid structures with the boundary must not create spurious reflections back into the flow.

A practical computational domain is constructed by imposing artificial boundaries around flows of interest. To establish boundaries at a distance where flow structures have effectively dissipated to the undisturbed state would often be computationally impractical, if not impossible. In many cases, the desired flow characteristics lie in a much more compact region. In order to accurately simulate the physical system, the boundaries must therefore allow flow structures to propagate out without reflection of nonphysical phenomena back into the domain.

In the case of aeroacoustic simulation, this is particularly important as sound computation is especially sensitive to spurious perturbations. The domain should only be large enough to



capture structures that significantly contribute to aeroacoustic sourcing. Dynamically significant flow structures may arise that are not of aeroacoustic interest, and could be present near the domain boundary. Such fluctuations could cause nonphysical wave reflection if boundary conditions are not carefully established. Such spurious waves can easily interfere with the calculation of acoustic propagation [11].

#### 2.3.4.1 Convection-Zone Boundary Conditions

In order to prescribe numerically transparent boundaries, the time-dependent solution must be fully defined along the domain edge. For external flows, this is not usually possible to do explicitly. For any transient hyperbolic system, such as the compressible Navier-Stokes equations in the inviscid limit, the behaviour along the boundary is described by outgoing waves as well as incoming waves from outside the domain. The outgoing waves are sufficiently defined by the interior solution, but the incoming waves have no such support. For a well posed problem, the precise behaviour of incoming waves must be determined, or at least sufficiently simulated [28].

Characteristic analysis of the one dimensional Euler equations [28] yields the eigenvalues

$$\lambda_1 = u - c, \lambda_2 = u, \lambda_3 = u + c,$$

which represent the propagation velocities of the characteristic waves, where  $u$  is the convective velocity and  $c$  is the speed of sound. Analogously applying these characteristics to the compressible Navier-Stokes equations, we examine them along an arbitrary boundary [24]. At an outflow boundary, we only want to pass internal structures out of the domain. For this boundary type,  $\lambda_2$  and  $\lambda_3$  are defined by the interior solution, and pose no direct complication to condition prescription. However, the incoming wave at velocity  $u - c$  must be somehow treated. At an inflow boundary, it may be desired to pass flow features into the domain while simultaneously preventing reflection of outgoing acoustic waves. For a well posed inflow, we find that there are several boundary condition sets that are well posed [24]. The problem with explicit definition of flow variables is that they are rigid and will cause nonphysical reflections.

Poinsot and Lele dealt with boundary difficulties by proposing evolutionary conditions that estimate desired inflow and outflow behaviours based on approximations of characteristic wave amplitude [24]. Another, mathematically simpler, method of approximating nonreflecting boundary treatments was proposed by Freund [11]. Freund's method is zonal-based, in that the computational domain is extended by a nonphysical buffer region. In this region, the flow is conditioned to provide nonreflecting support for the physical domain. The compressible Navier-Stokes equations are modified with the addition of two nonphysical terms within the boundary region [11],

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_j)}{\partial x_j} + U(x_1) \frac{\partial \rho}{\partial x_1} = -\sigma(x_1)(\rho - \rho_0) \quad (2.45)$$

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_i u_j)}{\partial x_j} + U(x_1) \frac{\partial(\rho u_i)}{\partial x_1} = \frac{\partial \tau_{ij}}{\partial x_j} - \sigma(x_1)[\rho u_i - (\rho u_i)_0] \quad (2.46)$$

$$\frac{\partial(\rho e)}{\partial t} + \frac{\partial(\rho e u_j)}{\partial x_j} + U(x_1) \frac{\partial(\rho e)}{\partial x_1} = \frac{\partial(u_j \tau_{ij})}{\partial x_j} - \sigma(x_1)[\rho e - (\rho e)_0], \quad (2.47)$$

for boundaries in the normal  $x_1$  direction.

In general, the artificial convection  $U(x_1)$  is chosen so that  $|U(x_1)| \geq c$ . The artificial damping term,  $\sigma$ , drives the solution along the boundaries to the quiescent state, supporting the numerical solution. The introduction of these artificial terms imposes numerical forcing in the boundary region, which causes spurious waves back into the physical domain. To ease this problem, a smooth transition in compact support from the physical domain to  $|U(x_1)| \geq c$  is required. Furthermore, the nonreflective efficacy of the buffer zone can be increased by increasing the size [37].

One of the strengths of Freund's method is that it can be applied to both the inflow and outflow boundaries. To eliminate the problem of unknown waves propagating inward at an outflow boundary,  $U(x_1)$  is defined with an outward convective velocity above  $c$ . Now all of the characteristic waves propagate with a net outward velocity, so that the boundary behavior is fully described by the solution interior to the domain. Furthermore,  $\sigma(x_1)$  serves to quickly dampen outgoing waves to prevent spurious reflections, which are caused by the numerical forcing induced by the addition of the artificial terms. Since  $U(x_1)$  is directed outwards, nonphysical reflections propagating inward from the boundary are slowed, spending longer time in the dissipative boundary zone [11].

On an inflow boundary,  $U(x_1)$  is chosen to be inward pointing. The effect of this is twofold.

Structures and desired flow characteristics that are passed into the domain propagate quickly through the nonphysical boundary zone, undergoing minimal artificial damping. Additionally, acoustic waves propagating upstream are asymptotically slowed and dissipated to the quiescent state [11].

### 2.3.5 Ffowcs-Williams Hawkins Analysis

While efficient computation favors a minimizing of the domain, the acoustic far-field and regions of interest can be quite large. In addition to spatial requirements, direct computation of aeroacoustics concurrent with the flow field places heavy computational demands alongside the difficulty of accurately resolving acoustic radiation scales. To circumvent these difficulties, hybrid methods are used, where an appropriate acoustic analogy is solved after the flow solution is computed in the near-field. The Ffowcs Williams Hawkins method has shown to be a popular approach. In theory, this amounts to solving the Ffowcs Williams Hawkins equation (2.25) upon a discrete flow solution. In practice, analysis of the Ffowcs Williams Hawkins equation (2.25) runs into several challenges.

In its exact form it contains a computationally expensive volume integral of the computationally expensive stress tensor  $T_{ij}$ . Furthermore, in two dimensions, the Green's function

$$G(\mathbf{x}, \mathbf{y}, t - \tau) = \frac{H(c_0(t - \tau) - |\mathbf{x} - \mathbf{y}|)}{2\pi c_0 \sqrt{c_0^2(t - \tau)^2 - |\mathbf{x} - \mathbf{y}|^2}} \quad (2.48)$$

necessitates a infinite time integral. While truncation of the time span can be used to approximate the solution, the timespan required could still pose computational difficulties [23].

An eloquent way around this is to perform the integration in the frequency domain and assume a periodic solution. A flow solution only needs to be run long enough to capture the longest frequencies of interest. For aperiodic flows, windowing of the solution can be used to approximate a periodic state. Algorithms for both two and three dimensional Ffowc Williams Hawkins integrations in the frequency domain were developed by Lockard and are readily available [23].

The presence of the volume integral can be approximated by appropriate placement of the Ffowcs Williams Hawkings surface  $S$ . While it is natural to consider a computational surface coinciding with the surface of solid obstacles in the flow, it is not required. Alternatively, a fully permeable surface in the flow may be prescribed. Placement of such a surface precludes the ability to predict the radiation field within the surface, but since far-field acoustics are primarily considered, this limitation is of little practical consequence. Placement of  $S$  is determinant on the flow in question. The volume integral across the stress tensor is performed outside of  $S$ , so in choosing  $S$  to fully encompass all significant shear stresses, equation (2.25) can be sufficiently approximated even with neglecting the volume integration. Though the surface integrals do not explicitly consider the quadrupole source, shear mixing effects within  $S$  are captured by the monopole and dipole terms. Furthermore, quadrupole terms are less efficient than other sources, so eliminating the volume integral can be done safely except in extended quadrupole dominated fields. Flows, like free jets, where there are no strict monopole or dipole terms, and the quadrupole field extends semi-infinitely, become problematic with these methods [23]. The flows considered here contain solid obstacles, and therefore contain the more efficient dipole source.

## Chapter 3

### Parallelization of Ray-Tracing Algorithm and Aeroacoustic Analogy

#### 3.1 Extension of CAD Model Analysis Algorithm to Parallel Computers

Geometry for complex obstacles can be efficiently generated using CAD software. In order to use the geometry with volume penalization methods, a masking function  $\chi(\mathbf{x})$  for a given grid must be constructed from the CAD files. Tracing rays emitted from grid points, and counting their intersections with the obstacle surfaces, effectively determines the local  $\chi$  value.

##### 3.1.1 Optimization of Ray-Tracing for Structured, Multi-Resolution Grid

Incorporation of the ray tracing algorithm into the adaptive solver must be done carefully to preserve computational efficiency. For the ray-tracing algorithm that is used with the AWC solver, not all nodes incur equal computational cost. First, when the geometry is initialized, a bounding box is created reflecting the extents of the geometry on the cardinal axes. For external flow regimes, this efficiently determines  $\chi$  for the large number of points that are not in the vicinity of the obstacle at a minimum cost.

At the next level, points that fall within the bounding box are considered. For rays parallel to the principle axes, root finding algorithms are used to determine the location of intersections. The computational cost is highly dependent upon the total number of intersections, the type of surface, and the nature of the geometry itself. This particular code classifies discrete geometry elements as either a revolved or non-uniform rational B-spline (NURBS) entity.

One practical difficulty with implementation of the algorithm is the representation of complex

geometry as a conglomeration of distinct entity. Contact surfaces between entities need to be rendered as internal points for the overall geometry. However, points on the surface can be rendered as external when the surface itself is counted as an intersection. When the grid adapts on the perceived surface, this causes the needless addition of large numbers of points along the internal seam between entities, as shown in Figure 3.1a. The geometry consists of a half-circle and half-ellipse joined along the minor axis. This effect is avoided by adding in a nominal offset smaller than the finest resolution of the grid. The discrete mask,  $\chi$ , is unaffected, but the internal seam is removed, as in Figure 3.1b.

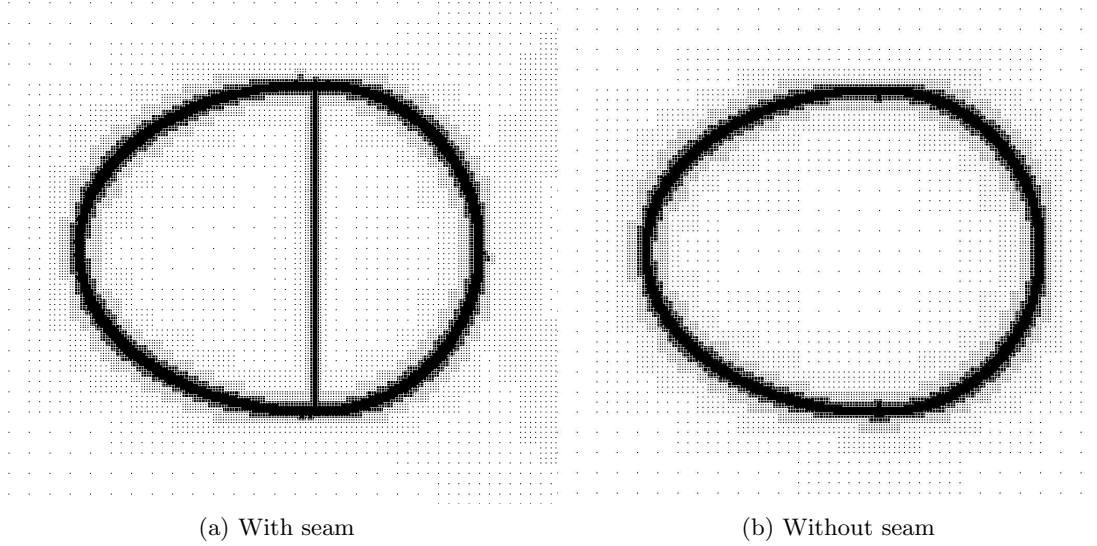


Figure 3.1: Grid adapted to solid obstacle comprised of elliptical and circular entities.

In general, each point with an intersecting ray will add appreciable computational cost to a calculation as the root-finding algorithms must converge to a precision that is smaller than the finest resolution of the grid. For large numbers of points, the relative computational cost can comprise a large portion of the computation time, especially if all nodes points must be re-evaluated at each time step.

For some solvers, such as AWCN, the structured nature of the grid can be a distinct advantage. For a fixed object, the masking function is static, so that the in/out data associated with

any given point does not change for the duration of the calculation. Unique points only need to be evaluated once. Much of the computational load for geometry analysis is therefore shifted to the grid adaptation on initial conditions, and only new points in the grid evaluated thereafter. The largest contributions to computational cost then come from highly transient structures that cause grid adaptation in the immediate vicinity of the obstacle.

### 3.1.2 Ray-Tracing for Parallel Computing

Parallelization of the ray-tracing algorithm proceeds in a straightforward fashion. The masking function,  $\chi$ , as well as a new-point identifier are attached to the point and communicated between processors as the solution is partitioned and repartitioned throughout the duration of a calculation. Once initialized, the geometry definitions are held in the memory, requiring only a single file I/O per processor. Though each point is evaluated independently, the parallelization of the code is highly problem specific. Subdomains containing highly transient solutions in the vicinity of computationally intensive geometry will incur a high cost, while static structures away from the obstacle are only marginally impacted.

### 3.1.3 Results and Discussion

The computational load of the ray-tracing algorithm is considered for a two dimensional acoustic scattering problem. The geometry consists of a cylinder located at the origin on a computational domain of  $\Omega = [-2, 3.5] \times [-2, 2]$ , and asymmetric about the y-axis. The cylinder is elliptical for  $x < 0$  and circular for  $x > 0$ . In the IGES format, it is represented by both recognized geometric entities: a revolved and a NURBS surfaces.

The propagation and scattering of an acoustic wave from a source at  $x = 2$  causes the grid to adapt continually for the duration of the calculation. The grid adapts in all regions of  $\Omega$ , including in the immediate vicinity of the obstacle. Though simple, this problem represents the broad spectra of difficulties for effective code parallization.

Figure 3.2a shows the total CPU time for the CAD analysis calculations at each timestep

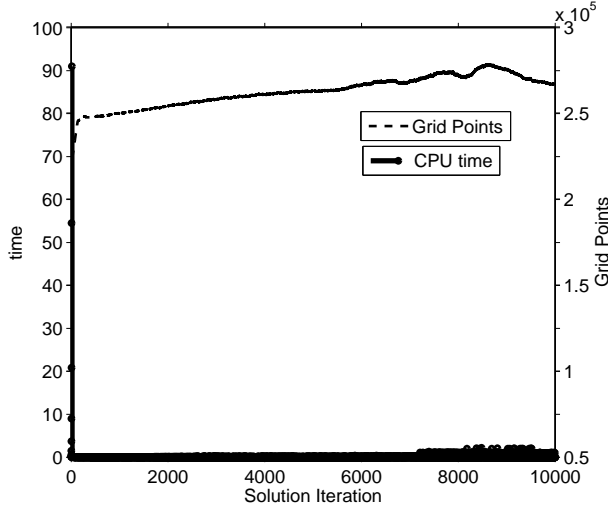


Figure 3.2: CPU time of CAD analysis and number of grid points at every timestep.

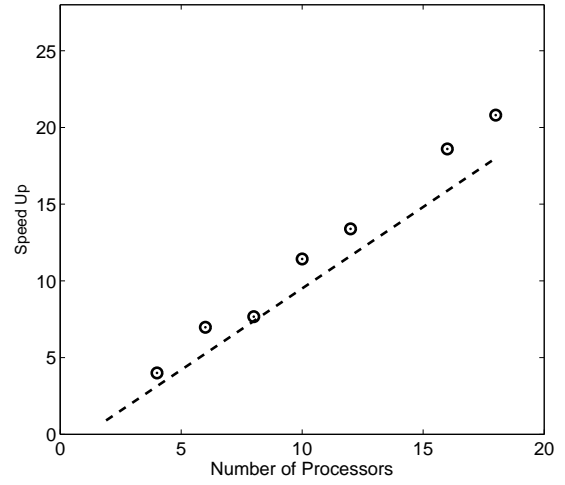


Figure 3.3: Strong scaling of CAD algorithm.

vis-à-vis the total number of grid points. A majority of the total walltime is incurred during the initializing of the grid. For the first several hundred iterations in time, the grid adapts heavily as the acoustic perturbation begins to propagate outward. However, since the points added are far away from the obstacle, the root finding algorithms are not used and the computational cost is negligible. When the acoustic waves approach the obstacle, near iteration 7000, ray-tracing is necessary to determine the mask for each point, and the computational cost rises accordingly. However, the number of points analyzed remains small.

The strong scaling for the ray-tracing algorithms in Figure 3.3a shows the embarrassingly parallel nature of this method for the problem considered.

### 3.2 Parallel Implementation of Ffowcs-Williams Hawkings Integration Surface

In order to calculate the aeroacoustic far-field, the time-accurate flow solution is needed for the Ffowcs Williams and Hawkings equations. The FWH code, developed by Lockard [23], requires constant time stepping and static nodes for the flow variables on the integration surface.



Since aeroacoustic analysis is performed in post-processing, interfacing with the AWC code only demands that appropriate solution files are produced and a strictly defined integration surface imposed in the computational grid.

### 3.2.1 Review of Serial Interface

The FWH code has previously been used with the AWC code for serial cases, and the interface is quite direct. The surface of integration for the FWH equations is defined by a discrete number of linear patches that are parallel to the principle axes. They are generated during the initialization of the code by adding points to the adaptive grid at a desired level of resolution. The most efficient patches have endpoints that lie on the  $j_{\min}$  grid level to avoid unnecessary adaptation. Once constructed, the variables are simply written at desired timesteps.

### 3.2.2 FWH parallel Interface

The parallel implementation of the FWH interfacing minimally affects the construction of the integration surface. During initialization, the FWH surface is defined on all processors, and each processor filters out points not resident to the local subdomain at each time step. At the desired timestep, all information for FWH points are passed to processor 0 for writing. In cases where it would be desirable to use an adaptive timestep, the parallel overhead is greater. Communication is required at every iteration in order to retain the integration surface in memory for interpolation.

### 3.2.3 Results and Discussion

Strong scaling is examined for two-dimensional acoustic scattering from an arbitrary solid obstacle located at the origin. The FWH surface is constructed so that it resides on multiple processors. A simple, first-order interpolation is used here for an adaptive, though higher-order schemes may be used with no additional communication requirements. The strong scaling for writing of FWH information is shown in Figure 3.4. The strong scaling across the low number of processors examined here is slightly less than 1, however, the adaptive timestep is a worst case

scenario and the low relative computational overhead makes this approach a feasible option.

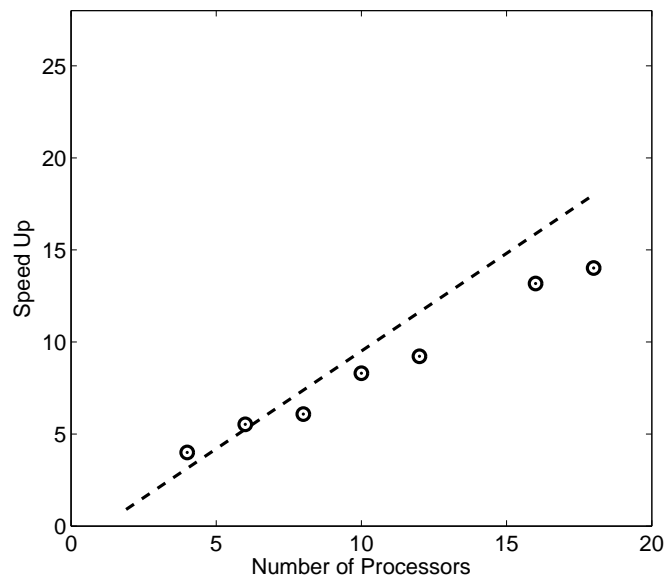


Figure 3.4: Strong scaling for writing of FWH integration surface to file.

## Chapter 4

### Characteristic-Based Volume Penalization Method

Immersed boundary methods have been developed to model flows around a solid obstacle without the need for body conformal meshes. While pseudo-conformal and multi-resolution grids are often used [2, 7, 13, 21], releasing the requirement for node/surface coincidence allows one to avoid costly re-meshing. This proposed method builds upon some of the principles that are employed for the Brinkman Penalization Method for compressible flows [21], particularly the treatment of Dirichlet boundary conditions.

While Brinkman penalization has shown to be effective in many fluid applications [2, 21], the drawbacks are pronounced. Steep gradients at the surface have shown to be problematic, requiring diffuse boundaries or, for Bae and Moon, local changes in differencing stencil [2]. Brinkman penalization is also limited to a small set of fluid problems and lacks generalizability.

The premise of this new Characteristic-Based Volume Penalization (CBVP) is to use Brinkman-style forcing for Dirichlet conditions, while introducing penalized convective terms to impose Neumann conditions. For Neumann conditions, the dominant penalization terms force the equations into a hyperbolic system with a single, inward-pointing characteristic. That way, the nonphysical penalized region only affects the physical solution through the derivative values at the boundary.

#### 4.1 Characteristic-Based Volume Penalization

Characteristic-Based Volume Penalization allows for the implementation of Dirichlet, Neumann, and Robin type boundary conditions by introducing forcing into the constitutive equations.

For a domain,  $\Omega$  containing obstacles  $O_i$ , desired boundary conditions are to be imposed upon the obstacle surfaces  $S_i$ . Penalization is applied only to the region inside of the obstacle by mean of a masking function,  $\chi$ , where

$$\chi = \begin{cases} 1 & \text{if } \mathbf{x} \in O_i, \\ 0 & \text{otherwise.} \end{cases}$$

The expression of penalized terms, and therefore the strict enforcement of the boundary conditions, is controlled by penalization parameter,  $\eta$ , in such a way that the solution converges as  $\eta \rightarrow 0$ .

Dirichlet conditions are imposed in the same fashion as with the Brinkman penalization method [21]. For a generalized quantity, the penalization term is added to the constitutive equation to provide forcing toward some target value,  $U_0$ , by

$$\frac{\partial u}{\partial t} = \dots \text{RHS} \dots - \frac{\chi}{\eta_b}(u - U_0), \quad (4.1)$$

where  $\partial u / \partial t$  is a general constitutive equation for some variable of interest.

Neumann boundary conditions can be imposed by introducing similar-style forcing on the normal derivative. For a target normal derivative  $q$  the governing equation for  $u$  is penalized

$$\frac{\partial u}{\partial t} = \dots \text{RHS} \dots - \frac{\chi}{\eta_c} n_k \left( \frac{\partial u}{\partial x_k} - q \right), \quad (4.2)$$

for a desired value,  $q$ , where  $n_k \hat{e}_k$  is the inward pointing surface normal of the obstacle. As  $\eta_c$  becomes small, the penalization term begins to dominate (4.2). While the right hand side of the constitutive equation is still present in the penalized region, it acts on a much larger timescale. For a sufficiently small penalization parameter, (4.2) becomes a hyperbolic equation,

$$\frac{\partial u}{\partial t} = -\frac{\chi}{\eta_c} n_k \left( \frac{\partial u}{\partial x_k} - q \right) \quad (4.3)$$

within the obstacle. For (4.3), there is a single characteristic pointing inward on the obstacle, and the steady state solution is

$$\frac{\partial u}{\partial x_k} = q. \quad (4.4)$$

The solution  $u$  on  $s_i$  is governed by the constitutive equations, while the nonphysical internal region only affects the fluid through the normal derivative. The characteristic ensures that the nonphysical internal solution does not propagate outward.

Note that these schemes (4.1-4.2) are generic and not applied to any particular system of PDE's. Penalization can be independently imposed for any set of problem specific quantities, and extended to the governing system provided that consistent relationships are available.

#### 4.1.1 Penalized Navier-Stokes Equations

For the problems considered here, penalization is prescribed for adiabatic, no-slip, and an approximate Neumann boundary condition, on immersed boundaries  $S_i$ . That is

$$\left. \begin{aligned} \frac{\partial \rho}{\partial n} &= 0, \\ \mathbf{u} &= \mathbf{U}_0, \\ \frac{\partial T}{\partial n} &= 0, \end{aligned} \right\} \text{ on } S_i \quad (4.5)$$

must be satisfied. While the Neumann condition on density is disingenuous considering the adiabatic condition, it is an approximation made to simplify and speed up the calculation. For the boundary layer of an adiabatic obstacle,

$$\frac{\partial \rho}{\partial n} = \frac{\rho}{p} \frac{\partial p}{\partial n} = O\left(\frac{1}{\sqrt{Re}}\right) \quad (4.6)$$

at the interface. This approximation works well for high Reynolds number flows.

To enforce this set of boundary conditions, equations are assumed of the form

$$\frac{\partial \rho}{\partial t} = \dots - \frac{\chi}{\eta_c} n_k \frac{\partial \rho}{\partial x_k}, \quad (4.7)$$

$$\frac{\partial \rho u_i}{\partial t} = \dots - \frac{\chi}{\eta_b} \rho (u_i - U_{0i}), \quad (4.8)$$

$$\frac{\partial T}{\partial t} = \dots - \frac{\chi}{\eta_c} n_k \frac{\partial T}{\partial x_k}. \quad (4.9)$$

Note that both (4.7) and (4.9) make use of the same parameter,  $\eta_c$ . This is following a simplest convention of penalizing like conditions similarly, as well as noting that pressure, density, and pressure are related through the equation of state. Using (4.7-4.9), the ideal gas equation, and the energy equation of state, the penalized nondimensional Navier-Stokes equations for boundary

conditions (4.5) are

$$\frac{\partial \rho^*}{\partial t^*} = -\frac{\partial \rho^* u_j^*}{\partial x_j^*} \boxed{-\frac{\chi}{\eta_c} n_k \frac{\partial \rho^*}{\partial x_k^*}}, \quad (4.10)$$

$$\frac{\partial \rho^* u_i^*}{\partial t^*} = -\frac{\partial(\rho^* u_i^* u_j^*)}{\partial x_j^*} - \frac{\partial p^*}{\partial x_i^*} + \frac{1}{Re_a} \frac{\partial \tau_{ij}^*}{\partial x_j^*} \boxed{-\frac{\chi}{\eta_b} \rho^* (u_i^* - U_{0i}^*)}, \quad (4.11)$$

$$\begin{aligned} \frac{\partial \rho^* e^*}{\partial t^*} = & -\frac{\partial}{\partial x_j^*} [(\rho^* e^* + p^*) u_j^*] + \frac{1}{Re_a} \frac{\partial(u_i^* \tau_{ij}^*)}{\partial x_j^*} + \frac{1}{(\gamma-1)} \frac{1}{Re_a Pr} \frac{\partial}{\partial x_j^*} \left( \mu \frac{\partial T^*}{\partial x_j^*} \right) \\ & \boxed{-\frac{\chi}{\eta_c} \frac{1}{(\gamma-1)} n_k \frac{\partial p^*}{\partial x_k^*} - \frac{\chi}{\eta_b} \rho^* u_i^* (u_i^* - U_{0i}^*) + \frac{\chi}{\eta_c} \frac{u_i^* u_i^*}{2} n_k \frac{\partial \rho^*}{\partial x_k^*}}, \end{aligned} \quad (4.12)$$

where it remains unchanged that

$$\begin{aligned} p^* &= \frac{\rho^* T^*}{\gamma}, \\ \tau_{ij}^* &= \mu^* \left( \frac{\partial u_i^*}{\partial x_j^*} + \frac{\partial u_j^*}{\partial x_i^*} - \frac{2}{3} \frac{\partial u_k^*}{\partial x_k^*} \delta_{ij} \right), \\ e^* &= \frac{1}{2} u_i^* u_i^* + c_p T^* - \frac{p^*}{\rho^*}. \end{aligned}$$

In a general sense, equations (4.10-4.12) are valid on all  $\Omega$ . However, strict application across the domain can lead to a convergence of characteristics with multi-valued singularities. This will invariable arise for any geometry where the surface normals converge within the computational domain, such as the typical case of a fully enclosed obstacle. Depending on the variables penalized in this manner, this can lead to accumulation in the regions of intersection. However, the region where the penalization terms affect the fluid solution is limited to a narrow region along the inside surface of the obstacle. For the adaptive wavelet solver used here, this region only consists of the points belonging to the differencing stencil for any fluid node. By properly resolving the obstacle, intersections can typically be avoided, except in the case of thin geometry. The penalization in (4.2) therefore only has local support within this narrow region, and effective sink terms prevent accumulations within the obstacle. Distance functions based on the masking term  $\chi$  have been used effectively to provid local support functions, though their practical construction is omitted here.

## 4.2 Stability and Artificial Viscosity

As noted by Liu and Vasilyev, the Brinkman term,  $\eta_b$ , introduces additional stiffness into the problem [21]. The solution in  $\rho \mathbf{u}$  is driven to the target value of  $U_0$  on the timescale of  $\eta_b$  within the obstacle, which is much smaller than the acoustic timescales of the problem. At the interface, this forcing creates a boundary layer within the obstacle. The boundary layer within the obstacle has a length-scale on the order of  $\delta = \sqrt{\frac{\eta_b}{Re_a}}$ . For stability, the boundary layer ought to be resolved, implying a mesh resolution of  $\Delta x_i < \sqrt{\frac{\eta_b}{Re_a}}$  at every resolution on the adaptive grid. For moderate values of  $\eta_b$ , the resolution requirements can quickly become computationally burdensome, even for adaptive multi-resolution methods.

Insufficiently resolved, the internal boundary layer leads to a discontinuity in the first derivative of momentum. This causes oscillations that propagate outward into the fluid domain. While these are low amplitude oscillations, the adaptive grid attempts to resolve them, greatly adding to the computational cost of the solution.

One solution would be to use a diffuse boundary region for the obstacle, where  $\chi$  transitions smoothly from 1 to 0 across a finite distance. Since a sharp interface becomes important at high Reynolds numbers, an alternative method is to use artificial viscosity to supplement physical viscosity at the solid-fluid interface. The length scale of a boundary layer,  $\delta$ , for a solution of timescale  $\tau$  with viscosity  $\nu$  is  $\delta^2 = \nu\tau$ ; so for a prescribed resolution, the viscosity  $\nu$  required to sufficiently smooth the solution inside the boundary layer is

$$\nu > \frac{\Delta x^2}{\eta_b}. \quad (4.13)$$

That is to say that  $\nu$ , the necessary viscosity for numerical stability, is dependant upon the desired resolution. For low Reynolds number flows or high resolution grids, this requirement will be met by the physical viscosity. Where this requirement is not met, artificial viscosity can be provided to compliment the physical viscosity through

$$\nu_n = \max \left\{ \alpha \frac{\Delta x^2}{\eta_b} - \frac{1}{Re_a}, 0 \right\}. \quad (4.14)$$

Since proper resolution requires  $O(10)$  points in the boundary layer, the factor  $\alpha$  is on the order of  $O(100)$ . While loosening the resolution requirement, the use of artificial viscosity contributes to additional error in the solution by lengthening the boundary layer within the obstacle.

### 4.3 Error Estimation From Acoustic Theory

The addition of the convective terms associated with penalization parameter  $\eta_c$  on the Navier-Stokes equations, (4.10-4.12), implies a discontinuous change in the speed of sound at the interface. From acoustic theory, this impedance mismatch would cause physical reflection of incoming waves based upon the relation. The reflection coefficient for incident pressure waves at an interface is

$$R = \frac{Z_2 - Z_1}{Z_2 + Z_1}, \quad (4.15)$$

where the impedance in terms of the density, cross-sectional area  $S$ , and speed of sound  $c$  is  $Z = \rho c/S$ . Since the density and cross-sectional area of the fluid for both regions is equal at the boundary,  $R$  is simply given by

$$R = \frac{c_2 - c_1}{c_2 + c_1} = \frac{1 - \eta_c}{1 + \eta_c} \approx 1 - 2\eta_c. \quad (4.16)$$

For reflected acoustic waves, the expected amplitude error is  $O(\eta_c)$ . However, it is worthwhile to note that does not consider the errors introduced from the penalized no-slip condition.

### 4.4 Asymptotic Analysis

While the penalized Navier-Stokes equations cannot be directly analyzed, rigorous asymptotic analysis of the acoustic timescale behavior gives insight into the error convergence of the penalization parameters  $\eta_b$  and  $\eta_c$ . One-dimensional flow is considered for an acoustic pulse reflecting off an obstacle. The error is examined in the fully reflected pulse.

For simplicity, a further modification is made to the penalized Navier-Stokes equations (4.10-4.12). The inviscid (Euler equation) terms are removed within the penalized region.



$$\frac{\partial \rho^*}{\partial t^*} = -\boxed{(1-\chi)} \frac{\partial \rho^* u_j^*}{\partial x_j^*} - \frac{\chi}{\eta_c} n_k \frac{\partial \rho^*}{\partial x_k^*}, \quad (4.17)$$

$$\frac{\partial \rho^* u_i^*}{\partial t^*} = -\boxed{(1-\chi)} \left[ \frac{\partial (\rho^* u_i^* u_j^*)}{\partial x_j^*} + \frac{\partial p^*}{\partial x_i^*} \right] + \frac{1}{Re_a} \frac{\partial \tau_{ij}^*}{\partial x_j^*} - \frac{\chi}{\eta_b} \rho^* (u_i^* - U_{0i}^*), \quad (4.18)$$

$$\begin{aligned} \frac{\partial \rho^* e^*}{\partial t^*} = & -\boxed{(1-\chi)} \frac{\partial}{\partial x_j^*} [(\rho^* e^* + p^*) u_j^*] + \frac{1}{Re_a} \frac{\partial (u_i^* \tau_{ij}^*)}{\partial x_j^*} + \frac{1}{(\gamma-1)} \frac{1}{Re_a Pr} \frac{\partial}{\partial x_j^*} \left( \mu \frac{\partial T^*}{\partial x_j^*} \right) \\ & - \frac{\chi}{\eta_c} \frac{1}{(\gamma-1)} n_k \frac{\partial p^*}{\partial x_k^*} - \frac{\chi}{\eta_b} \rho^* u_i^* (u_i^* - U_{0i}^*) + \frac{\chi}{\eta_c} \frac{u_i^* u_i^*}{2} n_k \frac{\partial \rho^*}{\partial x_k^*}. \end{aligned} \quad (4.19)$$

It is important to retain the viscous terms everywhere on  $\Omega$  for stability. Removing viscous terms would create a discontinuity from the Brinkman penalization forcing for the Dirichlet condition (4.1). This reduces or eliminates the need for artificial viscosity required to stabilize the penalization method.

#### 4.4.1 Asymptotic Analysis of the Fluid Region

First the amplitudes of the variables are asymptotically expanded for the smaller of  $(\eta_b, \eta_c)$ . The resulting equations, after substituting into the constitutive equations and discarding higher order terms, are the same for the cases where  $\eta_b \ll \eta_c$  and  $\eta_b \gg \eta_c$ . To preserve this generality,  $\eta = \min(\eta_b, \eta_c)$  for the following expansions.

For the region occupied by the fluid, the leading amplitude perturbation terms for nondimensionalized flow variables are

$$\rho_f(x, t) = 1 + \epsilon \rho'_{0f} + \epsilon \eta \rho'_{1f} + \dots, \quad u_f(x, t) = \epsilon u'_{0f} + \epsilon \eta u'_{1f} + \dots, \quad (4.20)$$

$$p_f(x, t) = \frac{1}{\gamma} + \epsilon p'_{0f} + \epsilon \eta p'_{1f} + \dots, \quad T_f(x, t) = 1 + \epsilon T'_{0f} + \epsilon \eta T'_{1f} + \dots \quad (4.21)$$

Substitution into (4.17-4.19) and retaining only first order terms yields the linear system

$$\frac{\partial u'_f}{\partial t} + \frac{\partial p'_f}{\partial x} = 0, \quad (4.22)$$

$$\frac{\partial p'_f}{\partial t} + \frac{\partial u'_f}{\partial x} = 0, \quad (4.23)$$

for both zero- and first-order perturbation quantities. Viscous terms are neglected in the high Reynolds number limit, where  $\frac{1}{Re_a} \ll \eta$ . Additionally, the relation  $\rho'_f = p'_f$  holds, and the flow

is isentropic. The resulting system (4.22-4.23) is simply the linear acoustic equations, describing small amplitude pulses propagating through the fluid.

#### 4.4.2 Asymptotic Analysis of the Penalized Region

Within the penalized region, representative of a solid obstacle, the asymptotic expansion of flow variable amplitudes are the same as within the fluid region. The principle difference enters in the value of the masking function,  $\chi = 1$  in equations (4.10-4.12). Once again, considering the case where  $\eta = \min(\eta_b, \eta_c)$ , the flow variables through the first-order perturbations is

$$\rho_p(x, t) = 1 + \epsilon \rho'_{0p} + \epsilon \eta \rho'_{1p} + \dots, \quad u_p(x, t) = \epsilon u'_{0p} + \epsilon \eta u'_{1p} + \dots, \quad (4.24)$$

$$p_p(x, t) = \frac{1}{\gamma} + \epsilon p'_{0p} + \epsilon \eta p'_{1p} + \dots, \quad T_p(x, t) = 1 + \epsilon T'_{0p} + \epsilon \eta T'_{1p} + \dots \quad (4.25)$$

Retaining first-order terms from (4.17-4.19) gives the following system in terms of the first-order perturbation quantities,

$$\frac{\partial \rho'_p}{\partial t} + \frac{1}{\eta_c} \frac{\partial \rho'_p}{\partial n} = 0, \quad (4.26)$$

$$\frac{\partial u'_p}{\partial t} + \frac{1}{\eta_b} u'_p = 0, \quad (4.27)$$

$$\frac{\partial p'_p}{\partial t} + \frac{1}{\eta_c} \frac{\partial p'_p}{\partial n} = 0, \quad (4.28)$$

where  $n$  indicates the inward-pointing normal. Again, the isentropic relation  $\rho'_f = p'_f$  holds. Here, the decision to remove inviscid terms from the Navier-Stokes equations is justified, as the equations are completely uncoupled within the penalized region, and as such, the analytical solution is much more straightforward.

The perturbation equations (4.26-4.28) form a linear hyperbolic system of PDEs with a single characteristic pointing inward on the penalized domain. Strong damping on velocity drives the solution towards the no-slip condition on timescale  $\eta_b$ , while strong convection on timescale  $\eta_c$  enforces the desired Neumann conditions. This implies  $\eta_b < \eta_c$  to avoid excessive phase errors. Otherwise, waves are convected inside the obstacle more quickly than the no-slip condition can be enforced, leading to a greater wave penetration past the interface.

In order to ensure continuity within the penalized domain, viscous terms from (4.18) must be retained. Assuming constant viscosity, (4.27) becomes

$$\frac{\partial u'_p}{\partial t} + \frac{1}{\eta_b} u'_p - \frac{1}{Re_a} \frac{\partial^2 u'_p}{\partial x^2} = 0. \quad (4.29)$$

#### 4.4.3 Asymptotic Analysis of Error Convergence

In order to examine the error convergence as  $\eta_b, \eta_c \rightarrow 0$ , the two systems (4.22-4.23) and (4.26-4.28) are solved on a one-dimensional split domain. Fluid occupies the semi-infinite region  $x < 0$ , while  $x \geq 0$  is considered the solid obstacle and appropriately penalized.

The solution in the fluid region, in terms of initial  $(u'_0(x), p'_0(x))$  conditions and boundary values  $(u'_1(t), p'_1(t))$  at the fluid-obstacle interface, is

$$u'_f(x, t) = \frac{1}{2} u'_0(x - t) + \frac{1}{2} p'_0(x - t) + \frac{1}{2} u'_1(x + t) - \frac{1}{2} p'_1(x + t) \quad (4.30)$$

$$p'_f(x, t) = \frac{1}{2} u'_0(x - t) + \frac{1}{2} p'_0(x - t) - \frac{1}{2} u'_1(x + t) + \frac{1}{2} p'_1(x + t) \quad (4.31)$$

for both zero- and first-order perturbations.

The solution in the penalized region can be determined for each variable independently. The first order perturbation of pressure is easily solved based on the propagation of boundary conditions for a single characteristic with speed  $\lambda = \frac{1}{\eta_c}$ ,

$$p'_p(x, t) = p'_1(t - \eta_c x). \quad (4.32)$$

The first order velocity perturbation of  $u'_p$  can be solved by transforming (4.29) into the form of an inhomogeneous heat equation through

$$w(x, t) = e^{-\frac{t}{\eta_b}} (u'(x, t) - u'(0, t)). \quad (4.33)$$

With continuity of the first derivative implied with the fluid domain, the solution of  $u'(x, t)$  on the semi-infinite domain is known [4]. Provided that the limits of the solution are finite and constant as  $x \rightarrow 0$  for variable parameter  $\eta_b$ , then the solution at the interface,  $u^*(t) = u'_p|_{x=0}$ , is  $O(\eta_b)$ . Now

considering the timescale  $t \gg \eta_b$ , the quasi-steady state solution to the boundary value problem (4.29) is found to be

$$u'_p(x, t) = u^*(t) \exp \left( -x \left( \frac{Re_a}{\eta_b} \right)^{1/2} \right). \quad (4.34)$$

Enforcing continuity and smoothness between the fluid domain and penalized domain solutions,  $u'_f|_{x=0} = u'_p|_{x=0}$  and  $\frac{\partial u'_f}{\partial x}|_{x=0} = \frac{\partial u'_p}{\partial x}|_{x=0}$ , yields the acoustic solution with leading error terms  $O(\eta_c, \eta_b^{1/2})$ . This reinforces that optimally, the penalization parameters should be chosen so that  $\eta_b < \eta_c$ .

## 4.5 Results and Discussion

The first benchmark problem is to verify the behavior of penalized equations for each of the Dirichlet and Neumann boundary conditions. One dimensional heat conduction transients allow for each boundary condition to be examined in the simplest case of a scalar valued solution without the effects of coupling.

Two general groups of fluid benchmark problems are considered to validate the penalization for no-slip, adiabatic boundary conditions on coupled equations. The first group is for acoustic scattering problems, and directly corresponds to the asymptotic analysis earlier in this chapter. A one dimensional case is used to show convergence of the solution on the penalization parameters. A second case shows a multi-dimensional application and the validity of curved boundaries. The second group of fluid simulations is formed around the canonical problem of external nonzero mean flow past a cylinder. Low Reynolds number, incompressible flow is used to validate the methods, especially for boundary layer separation, under the presence of a stabilizing viscous zone at the surface. Periodic vortex shedding is also considered to validate the methods for compressible flows at moderate Reynolds numbers.

All problems are solved using Freund-type boundary conditions in a zone bounding the physical domain. The characteristic length for two dimensional problems is the diameter of the cylinder.

#### 4.5.1 Benchmark Problem I: One Dimensional Heat Conduction

Heat conduction provides a good platform for examining CBVP of each boundary condition individually. There is a corresponding physical application for each boundary condition, and furthermore, these problems exemplify the versatility of the method. Unlike earlier Brinkman methods, CBVP is not limited to fluid problems, but is rather a general form of individual boundary conditions that can be applied to a wide variety of PDEs.

The constitutive equation is the one-dimensional heat equation for nondimensional temperature perturbation  $T$ ,

$$\frac{\partial T}{\partial t} - k \frac{\partial^2 T}{\partial x^2} = 0, \quad (4.35)$$

and is applied in the physical domain. The physical domain consists of  $\Omega = [-1, 0]$ , while the penalized domain is  $\Omega_p = [0, 0.5]$ . For all cases, the initial conditions are  $T_0(-1) = 1$  and  $T_0(x) = 0$  elsewhere. The left boundary condition is  $T(0, t) = 1$ .

This problem is solved for a nonadaptive grid. In determining the error, the solution is compared with numerical results for the exact boundary conditions imposed in the usual fashion at the right physical boundary,  $x = 0$ . The resolution used for all simulations is  $\Delta x = \frac{1}{1024}$ . Errors are examined during the transient solution for a thermal conductivity of  $k = 0.5$ .

Penalization for the Dirichlet condition is prescribed by (4.1) for constant temperature  $T(0, t) = 0$ . In Figure 4.1a, the error can clearly be seen converging on  $O(\eta_b^{1/2})$ . This agrees with the expected error term introduced by the no-slip condition on the asymptotic analysis.

The Neumann condition for an adiabatic boundary,  $\frac{\partial T}{\partial n} = 0$ , is imposed by penalization of the form in (4.2). The error converges on  $O(\eta_c)$ , shown in Figure 4.1b, and agrees with the error introduced by the Neumann boundary condition from the asymptotic analysis.

#### 4.5.2 Benchmark Problem II: One Dimensional Normal Wave

The penalized equations are solved on a domain  $\Omega = [-0.5, 0.5]$ , where the fluid occupies  $\Omega_f = [-0.5, 0]$ , and the solid obstacle is  $\Omega_s = [0, 0.5]$ . Initial conditions for the incident wave are

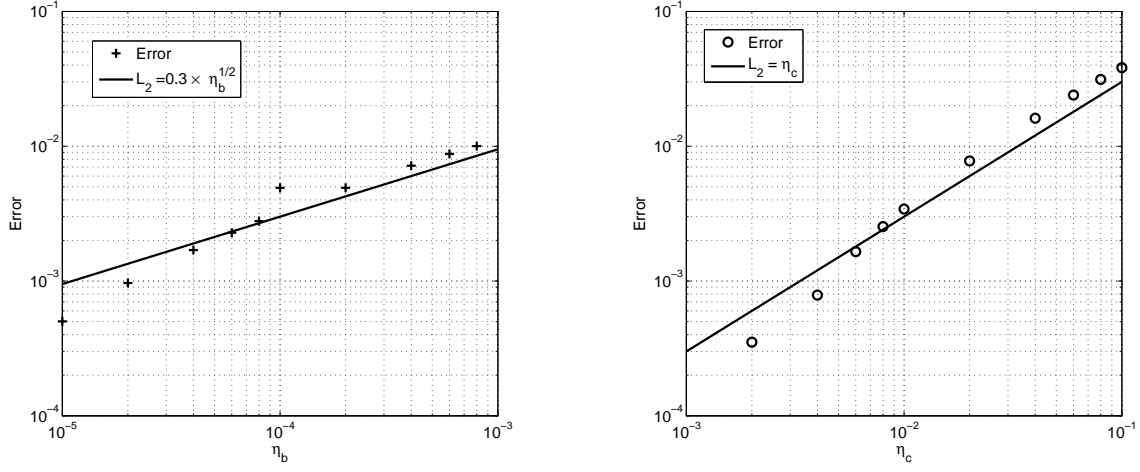


Figure 4.1: Time-averaged  $L_2$  norm error for converging  $\eta_c$  and  $\eta_b$ .

given by Gaussian distributions in the perturbation quantities

$$\rho' = u' = 10^{-3} \exp \left[ -\ln(2) \left( \frac{(x + 0.25)^2}{0.004} \right) \right], \quad (4.36)$$

with

$$e = \frac{1}{2} u_i u_i + c_p T - \frac{p}{\rho}. \quad (4.37)$$

The Reynolds number is  $Re_a = 10^{-5}$ .

This problem is first solved for the penalization scheme in (4.17-4.19) where the inviscid fluxes are removed and the penalized obstacle becomes uncoupled for  $Re \rightarrow \infty$ . This simplification corresponds to the equations used in the asymptotic analysis outlined in the previous section, and therefore error convergence of  $O(\eta_c, \eta_b^{1/2})$  is expected. In the second set, where the Navier-Stokes equations are left intact within  $\Omega_p$ , the problem is considered for both high-resolution cases where the stability requirement (4.13) is met, and a computationally efficient resolution where localized viscosity is introduced.

Figure 4.2 shows a fully reflected wave for high resolution and artificial viscosity cases. Note the increased phase and amplitude errors introduced by the presence of the viscous layer for waves travelling in the negative  $x$ -direction. The more pronounced phase error arises from the increased

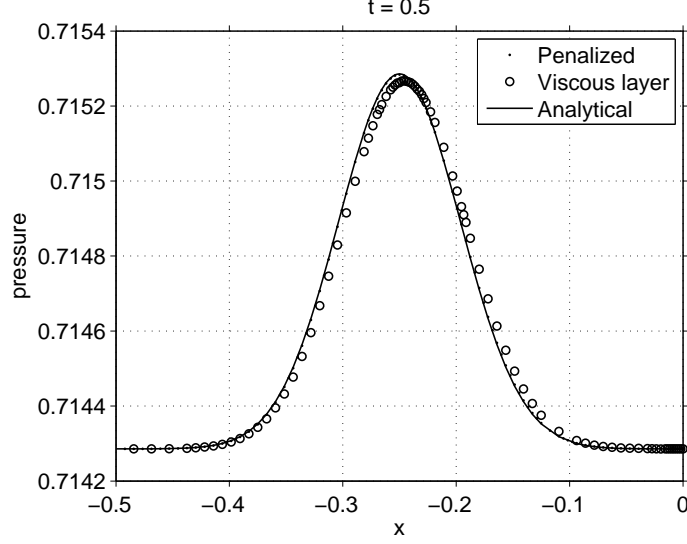


Figure 4.2: Fully reflected pressure waves for a high resolution penalized case, a case with artificial viscosity added near the solid interface, and the analytical solution.

offset stagnation point from the solid interface.

Where  $\eta_b = 10^{-3}$ , the highest effective resolution on the adaptive grid is  $\Delta x = O(10^{-6})$ , enough to resolve transient structures created by the Brinkman forcing term. For comparison, sufficient resolution of the boundary layer is  $O(\frac{1}{\sqrt{Re_a}})$ , or  $O(10^{-3})$ .

Figure 4.3a show the error convergence on a fully reflected pulse for the cases where  $\eta_b = \eta_c$  and  $\eta_b \gg \eta_c = 8 \times 10^{-4}$ . The error convergence of both cases is the same, namely  $O(\eta_b^{1/2})$ , implying that the error introduced  $\eta_c$  is relatively lower than that of  $\eta_b$ . This result matches well to the asymptotic analysis presented in the previous section. For  $\eta_c \gg \eta_b = 4 \times 10^{-5}$ , the error was examined for a pulse in mid reflection, and the  $L_2$  norm error convergence shown in Figure 4.3b. The error converges on  $\eta_c^{1/2}$  before quickly saturating from numerical viscosity. Note the disparate magnitudes of error incurred for cases where  $\eta_b \approx \eta_c$ . The value of  $\eta_b$  must be several orders of magnitude smaller than  $\eta_c$  in order to introduce comparable errors.

This result is beneficial for computational efficiency. When using Krylov-space time integration, the penalized equations have been found to be unstable in for  $\eta_c$ -based CFL that is much larger than  $O(20)$ . Satisfactory accuracy can still be obtained for  $\eta_c = 0.1$ , which allows for use of

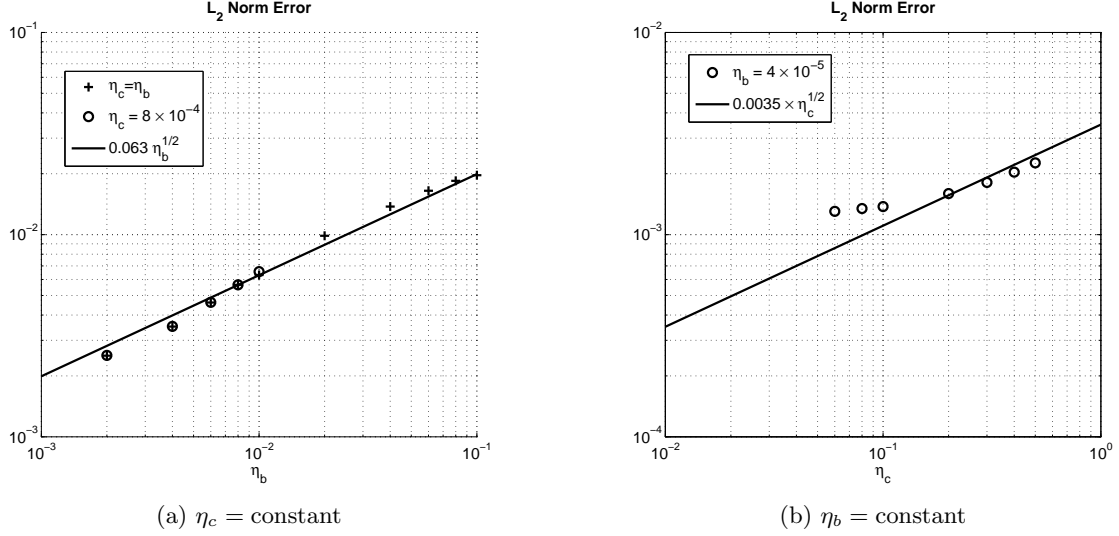


Figure 4.3: Time-averaged  $L_2$  norm error for converging  $\eta_c$  and  $\eta_b$  with Euler terms removed.

the acoustic CFL without implementing split timesteps.

The same problem was solved with the full Navier-Stokes equations left intact within the penalized region. Thus the equations remain coupled throughout the entire domain, even in the inviscid limit.

For  $\eta_c = \text{constant}$ , Figure 4.4a shows error convergence on  $O(\eta_b)$ . Conversely, where  $\eta_b = \text{constant}$ , Figure 4.4b shows initial error convergence on  $O(\eta_c^{0.85})$ , before saturating on some unknown error source. The translation in convergence plots for  $\eta_c = \text{constant}$  and  $\eta_b = \text{constant}$ , in Figures 4.4a and 4.4b respectively, indicate general error convergence on  $O(\eta_b(\eta_c + E)^{0.85})$ , where  $E$  is the undetermined error term from Figure 4.4b. Again, we see that  $\eta_b$  ought to be the more strictly controlled parameter.

For lower resolution cases, where the viscosity requirement in (4.13) is not met, artificial viscosity was added to a narrow region in the vicinity of the solid interface. While the theoretical thickness of the viscous layer must be on  $O(\eta_b)$ , the practical implementation is much thicker. A smooth support function is desired to avoid discontinuities in the solution derivative. In order to properly resolve the diffuse boundary between the physical viscosity and the artificial viscosity, the zone is about three times thicker than the theoretical requirement.



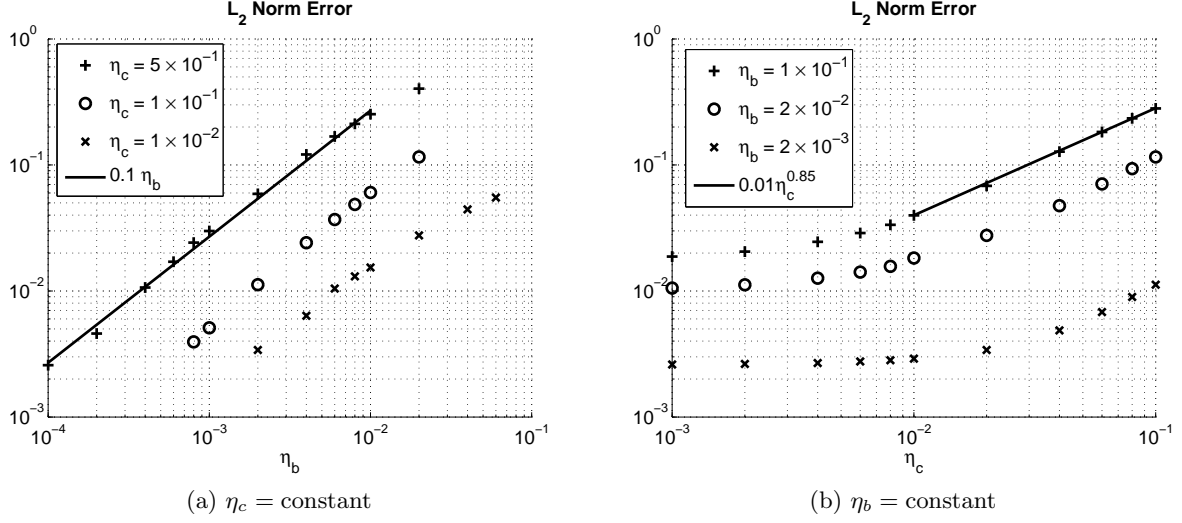


Figure 4.4: Time-averaged  $L_2$  norm error for converging  $\eta_c$  and  $\eta_b$ .

With use of artificial viscosity, the error quickly saturates for a given resolution, as shown in Figure (4.5). Further convergence of  $\eta_b$  increases the viscosity requirement, and any accuracy gains are offset. Therefore, in such cases, the accuracy is controlled by the resolution.

#### 4.5.3 Benchmark Problem III: Acoustic Scattering of a Single Source

Acoustic scattering from a two-dimensional cylinder is considered. This problem demonstrates multidimensional application of the penalization method, where the surface normal is different from the incidence angle of reflected waves. Furthermore, it shows the ability to resolve and properly model the effects of a curved surface. A cylinder of  $r = 0.5$  is located at the origin, and the initial conditions are a Gaussian distribution in pressure, given by the perturbation

$$p' = 10^{-3} \exp \left[ -\ln(2) \left( \frac{(x-4)^2 + y^2}{0.2^2} \right) \right]. \quad (4.38)$$

The initial conditions on the native variables are then  $\rho = \rho_0 + p'$ ,  $\rho u_1 = \rho u_2 = 0$  and  $e = c_p T - \frac{p}{\rho}$ .

The problem is solved for  $Re_a = 10^5$  on a physical domain  $\Omega_P = [-3, 5] \times [-3, 3]$ . The adaptive grid has 8 levels of resolution, ranging from  $[16 \times 12]$  to  $[2048 \times 1536]$ . This resolution is not sufficient for the stability of the penalization terms at the solid interface, so artificial viscosity

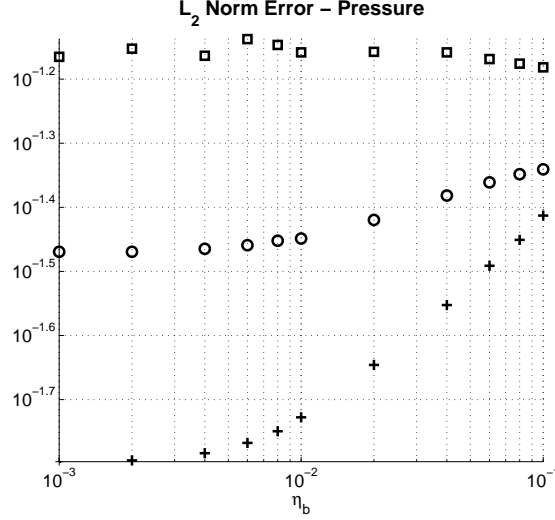


Figure 4.5: Error convergence for 1D acoustic reflection with artificial viscosity introduced at the solid interface.

has been introduced. The penalization parameters are set to moderate values of  $\eta_b = \eta_c = 2 \times 10^{-2}$  to reduce the size and strength of the viscous zone and to aid in a rapid calculation. The viscous zone about the obstacle has a width of  $O(D10^{-2})$ , where  $D$  is the cylinder diameter.

The computational domain is shown in Figure 4.6. The Fruend buffer zone is clearly shown, as well as the integration surface for the Ffowcs Williams and Hawkings equation. To include all major acoustic sources, the FWH surface encompasses both the initial acoustic perturbation and the obstacle. The solution in time is examined at mid- and far-field locations A-F in Figure 4.7.

The mid-field numerical solutions (Locations A-E) correspond well with the inviscid analytical solution. Some of the error arises from the curved surfaces of the obstacle being represented discretely on a rectilinear grid. Furthermore, the increased viscosity in the immediate vicinity of the obstacle dampens the wave reflection, introducing amplitude and phase errors. However, in light of the reduced computational costs, the results are quite satisfactory.

The far-field time history at Location  $F = (10, 10)$  lies outside of the computational domain. The numerical solution is only determined here by the use of the FWH analogy. The advantage of this method is clearly exemplified, as an accurate solution has been obtained while limiting the computation to aeroacoustic source regions.

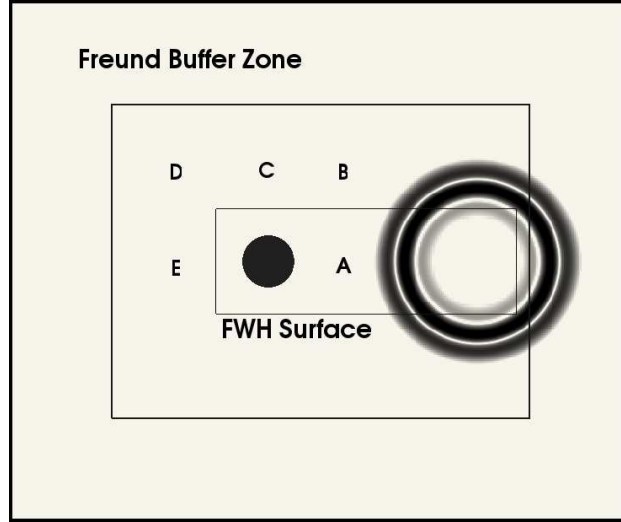


Figure 4.6: Layout of 2D acoustic scattering problem. A solid obstacle is located at the origin, and pressure waves propagate from a source centered at  $(4, 0)$ .

At many of the locations, the FWH solution is more accurate than the direct numerical calculation. This arises primarily from spurious reflections from the boundary. Location D, near the corner of the domain, is particularly prone to nonphysical reflections. Since the FWH surface is located further from the domain edge than most of the points, and does not encompass the sources of the reflections (i.e. the Freund buffer zone), these reflections have a much lower impact on the FWH solution. While a larger domain and buffer zone may be used to reduce spurious reflections in regions of interest, the use of aeroacoustic analogy obviates the need for this extra computational cost.

#### 4.5.4 Benchmark Problem IV: Incompressible Flow Past a Circular Cylinder

In order to verify the use of artificial viscosity along the immersed boundary, incompressible flow past a circular cylinder is considered. As additional viscosity is required to stabilize the solution at more moderate resolutions, it is important to consider its effect upon the boundary layer. For low Reynolds numbers, the boundary layer behavior in the wake of a circular cylinder is well documented for both experimental and numerical cases [5, 8, 10, 20, 29]

The flow is modelled as weakly compressible with  $M = 0.03$  and freestream Reynolds number

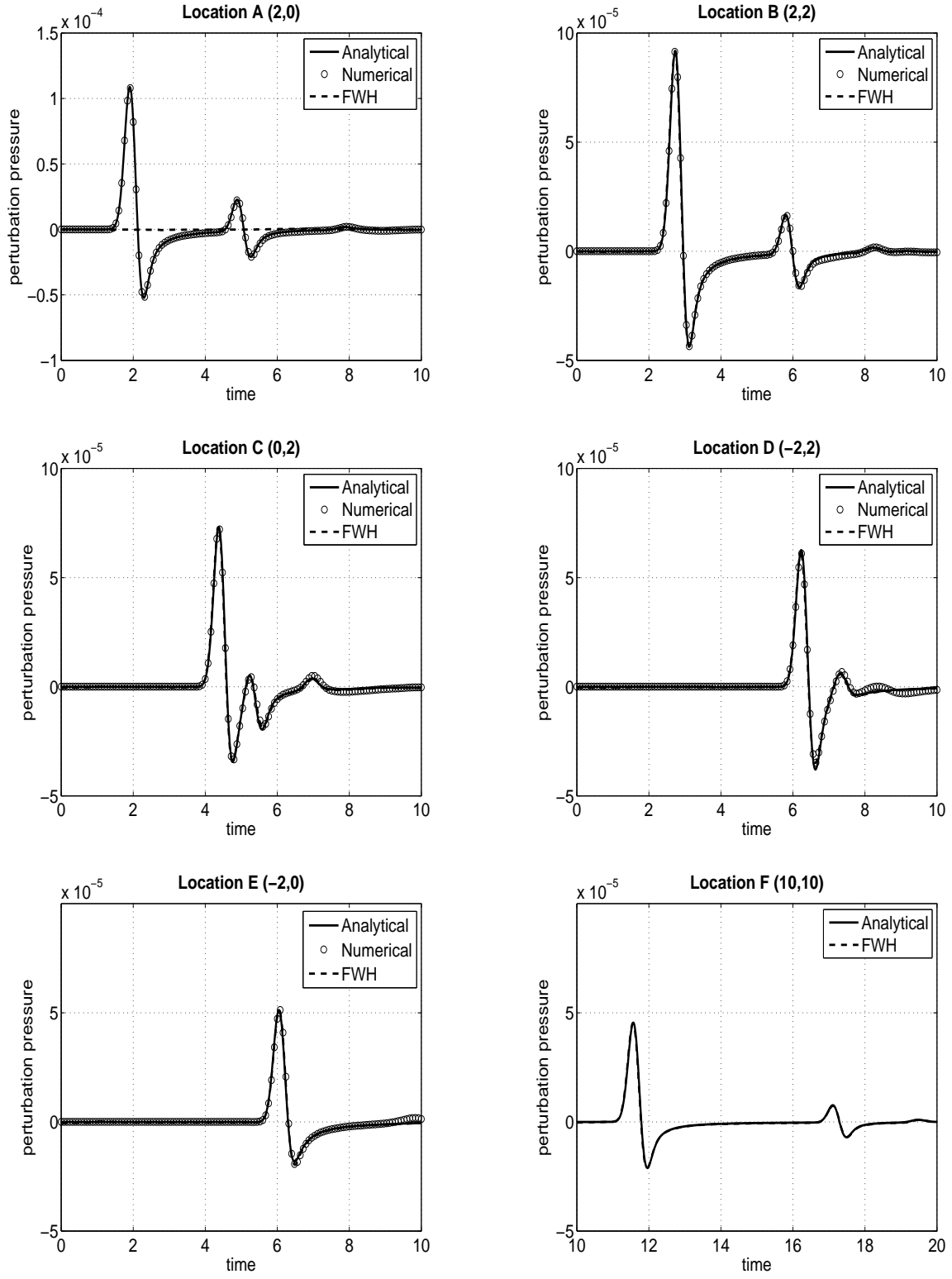


Figure 4.7: Time histories of pressure perturbations at several locations.

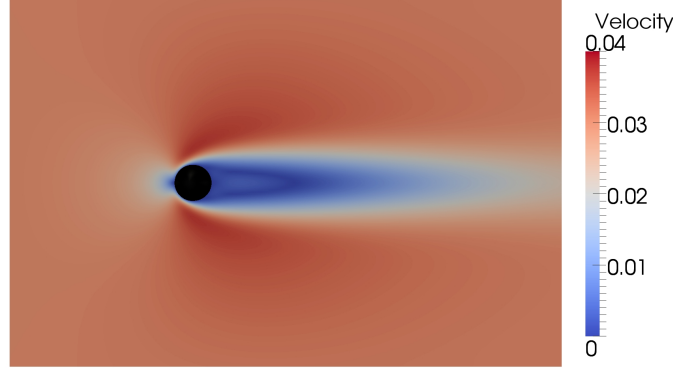


Figure 4.8: Velocity magnitude of weakly compressible flow past a cylinder at  $Re = 40$ .

of  $Re = 40$  ( $Re_a = 1333$ ). A cylinder of  $r = 0.5$  ( $D = 1.0$ ) is centered on the origin of domain  $\Omega_P = [-5, 10] \times [-5, 5]$ , and the penalization parameters are  $\eta_b = 2 \times 10^{-2}$  and  $\eta_c = 10^{-1}$ . The maximum effective resolution is  $2560 \times 3840$ , requiring additional viscosity to stabilize the penalization terms in the solution. The boundary layer separation point  $\theta$ , and the wake length  $L$  are both measured from the trailing edge of the cylinder.

The boundary layer separation point is highly dependent upon the Reynolds number [5]. At the resolution used, the effective freestream Reynolds number near the interface is  $Re \approx 10$ . From Coutanceau's results, the corresponding separation point would be  $\theta = 35^\circ$ .

The velocity of the flow field at steady state is shown in Figure 4.8. The location of the boundary layer separation in this problem was found to be  $\theta = 49.9^\circ$ . This is markedly less than numerical results found in the literature, enumerated in Table 4.1. The artificial viscosity delays boundary layer separation. For problems that are highly sensitive to separation locations, the use of artificial viscosity ought to be minimized by the use of high resolutions.

	$\Theta$	$L$	$C_D$
Dennis and Chang [8]	$53.8^\circ$	2.35	1.52
Fornberg [10]	$55.6^\circ$	2.24	1.50
Tullio et al. [7]	$53.7^\circ$	2.23	1.49
<b>Present</b>	<b><math>49.9^\circ</math></b>	<b>2.22</b>	<b>1.52</b>

Table 4.1: Numerical results for incompressible flow past a 2D cylinder at  $Re = 40$

Even with this discrepancy, the wake length and drag coefficient agree well with previously established results. The symmetrical separation region is  $L = 2.22$ , and the drag coefficient is  $C_D = 1.52$ . Compare this with the results in Table 4.1.

#### 4.5.5 Benchmark Problem V: Vortex Shedding from a Circular Cylinder

This problem considers impulsively started flow of  $M = 0.2$  and  $Re_a = 750$  ( $Re = 150$ ) on the physical domain  $\Omega_P = [-5, 10] \times [-5, 5]$ . A solid cylinder of  $r = 0.5$  is located at the origin. For CBVP, penalization parameters of  $\eta_b = 2 \times 10^{-2}$  and  $\eta_c = 10^{-1}$  are used. The maximum effective resolution is  $2560 \times 3840$ , requiring a viscous zone with effective Reynolds number  $Re \approx 10$  along the solid interface to satisfy requirement from equation 4.13.

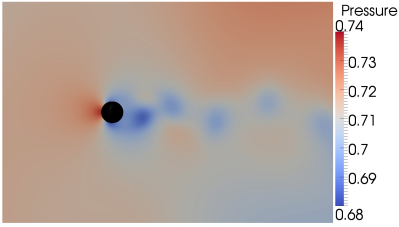


Figure 4.9: Pressure

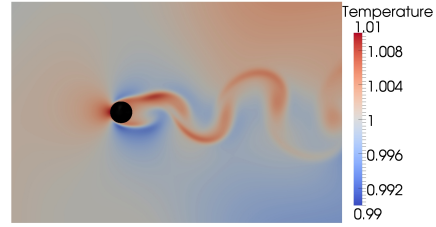


Figure 4.10: Temperature

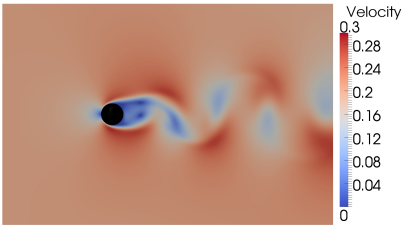


Figure 4.11: Velocity Magnitude

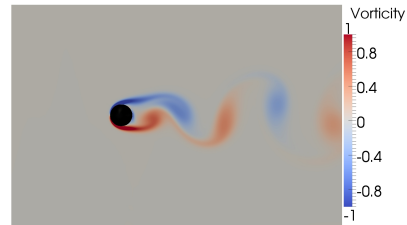


Figure 4.12: Vorticity

The instantaneous values of several flow variables are shown in Figures (4.9-4.12a), where the Karman vortex street is clearly seen. Along the surface of the cylinder, the flow remains unsteady throughout the duration of the calculation. Unlike previous simulations, which approached steady

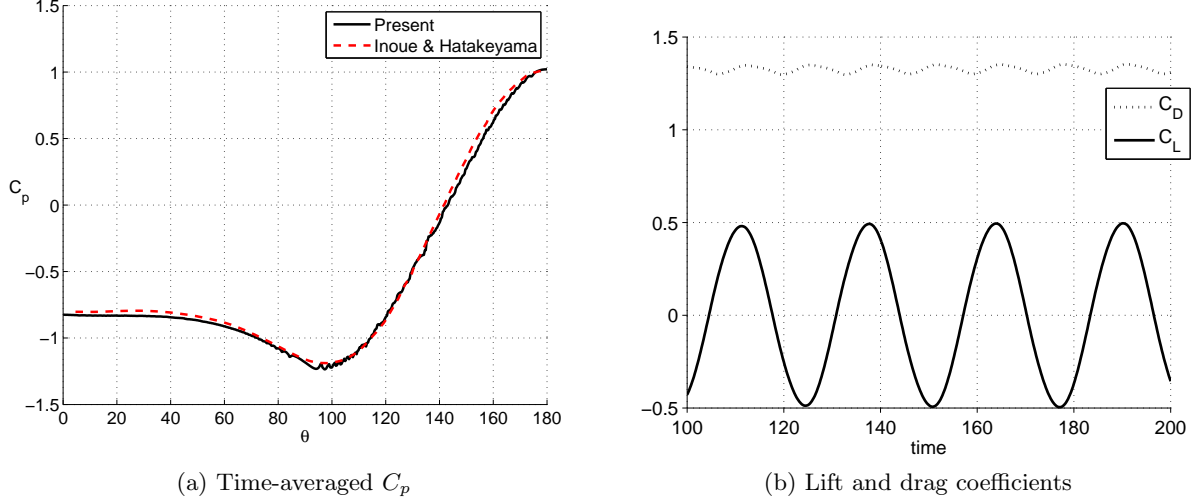


Figure 4.13: Force coefficients at  $Re = 150$

state along the solid interface, this demonstrates the stability and accuracy of CBVP for continually transient solutions. Slight nonphysical reflections can be seen from the outflow boundary where vortices exit the domain.

The time-averaged pressure coefficient,  $C_p$ , is shown in Figure 4.13a, normalized to the leading edge stagnation point. Note that  $\theta = 0^\circ$  corresponds with the trailing edge of the cylinder. The pressure coefficient corresponds well with the results of Inoue and Hatakeyama [16], which was solved using a cylindrical coordinate system. The noise in the present results is an artifact from representing a circle on a rectilinear grid and the accompanying interpolation of surface points. The artificial viscosity used to stabilize the penalization method does not appear to introduce much error into the pressure at the interface, an important result for determining aerodynamic forces on the surface.

Time variant force coefficients are shown in Figure 4.13b. This forcing suggests perpendicular acoustic dipoles with the lifting dipole being the dominant source. The present simulation corresponds well with the forcing results of Inoue and Hatakeyama [16], though with a slightly lower amplitude for  $C_l$ . The Strouhal number for the shedding frequency is  $St = 0.19$ , which agrees well with both numerical [16] and analytical [12] results, both of which indicate a Strouhal number of

$St \cong 0.19$  for  $Re = 150$ .

Since the physical domain does not fully encompass the wake, strong vortices are convected out of the  $x$  max boundary. Therefore, quadrupole sources from the stress in the vortices are lost from the aeroacoustic solution. However, dipole tones dominate the solution at such a low Reynolds number [23]. For integration of the FWH equation in the frequency domain, a combined Hanning/Dirichlet windowing function is used, with the Dirichlet function covering 75% of the signal sample. Lockard's results show a slight loss of acoustic amplitude due to the convection of vortices out of the FWH integration surface, though the tonal dominance remains intact [23].

Cross-stream from the obstacle ( $\theta = 90^\circ$ ), the aeroacoustic field is dominated by the shedding frequency caused by the lifting forces, while the dominant tone in the upstream/downstream directions ( $\theta = 0^\circ, 180^\circ$ ) arise from the forcing from drag. The pressure power spectra in Figure 4.14 shows the same amplitude and frequency relationship between the drag tones and lifting tones as the unsteady forcing in Figure 4.13b. The directionality of the harmonic tones indicates that the aeroacoustic dipoles have been modelled correctly radiate accordingly into the far-field. Noise at higher frequencies is an artifact of the low temporal resolution and spurious reflections at the downstream boundary.



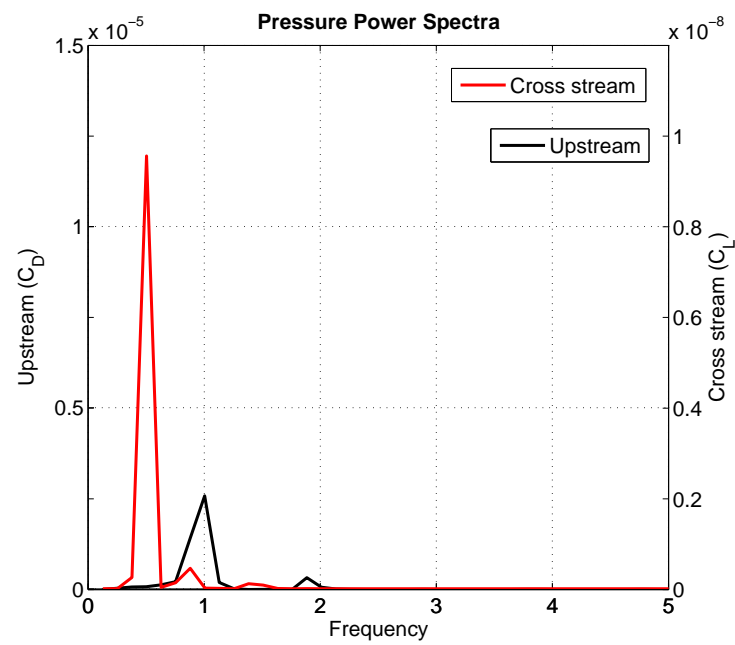


Figure 4.14: Pressure power spectra for vortex shedding at  $Re = 150$ .

## Chapter 5

### Conclusions and Future Research

#### 5.1 Conclusions

A cohesive model has been developed here for the accurate and efficient prediction of aeroacoustic radiation on parallel processors. The use of aeroacoustic analogy and immersed boundary method allows for the use of small, structured grids, controlling the problem size.

One of the principle outcomes is the development of a new penalization method for immersed boundaries where individual conditions can be imposed naturally. For viscous flow and acoustic propagation, this method has shown to be accurate with a rigorously controllable error. The results on a series of benchmark problems agree well with previously established results, validating the method on the acoustic timescale. Furthermore, this penalization method is sufficient for modelling aeroacoustic noise from the presence of solid obstacles. Owing to the general nature of the immersed boundary conditions, this volume penalization method can be applied beyond fluid dynamics to a wide variety of physical models.

One of the principle difficulties is the CFL limit that is imposed by the convective penalization term,  $\eta_c$ . For small  $\eta_c$ , instabilities result for problems solved on the acoustic timescale using Krylov-space time integration. This is mitigated through the use of upwind-differencing in penalized regions, as well as the greater error introduced by  $\eta_b$  terms.

## 5.2 Future Research

While the ability to use CAD generated geometry to define penalization masking functions is a powerful asset for Brinkman penalization, it is insufficient for the volume penalization method. Since CVPM also requires distance functions and surface normals, extension of IGES file analysis would be a worthwhile platform for further development.

Characteristic-based volume penalization can also be extended into problems of higher complexity, including supersonic and three-dimensional domains and problems with moving obstacles. Such cases would provide a comprehensive foundation for the use of CBVP with the Navier-Stokes equations.

Additionally, CBVP has only been used here for low Reynolds number viscous flows, where the Navier-Stokes equations can be solved directly. Many aeroacoustic flows of interest occur at high Reynolds numbers, requiring turbulence modelling. Further work can be done to apply CBVP to turbulence equations, including Reynolds-Averaged Navier-Stokes (RANS) and Large Eddy Simulation (LES).

One of the advantages to CBVP is the ability to implement general Neumann boundary conditions. The use of this method on problems involving heat transfer at a solid interface are yet unexplored.

Lastly, the CFL limit imposed by the  $\eta_c$  convective term can become problematic. Circumventing this problem using alternative time integration methods, or a split time step is worth further investigation for developing the practical usefulness of CBVP.

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